



Mathematics Education:
How to solve it?



**Proceedings of the
40th Conference of the International
Group for the Psychology of Mathematics Education**

Editors

Csaba Csíkos

Attila Rausch

Judit Szitányi



PME40, Szeged, Hungary, 3–7 August, 2016



Cite as:

Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4. Szeged, Hungary: PME.

Website: <http://pme40.hu>

The proceedings are also available via <http://www.igpme.org>

Publisher:

International Group for the Psychology of Mathematics Education

Copyrights © 2016 left to the authors

All rights reserved

ISSN 0771-100X

ISBN 978-1-365-46345-7

Logo: Lóránt Ragó

Composition of Proceedings: Edit Börcsökkné Soós

Printed in Hungary
Innovariant Nyomdaipari Kft., Algyő
www.innovariant.hu

TABLE OF CONTENTS

VOLUME 4 — RESEARCH REPORTS (OST – Z)

Osta, Iman; Thabet, Najwa	3–10
ALTERNATIVE CONCEPTIONS OF LIMIT OF FUNCTION HELD BY LEBANESE SECONDARY SCHOOL STUDENTS	
Otaki, Koji; Miyakawa, Takeshi; Hamanaka, Hiroaki	11–18
PROVING ACTIVITIES IN INQUIRIES USING THE INTERNET	
Ottinger, Sarah; Kollar, Ingo; Ufer, Stefan	19–26
CONTENT AND FORM – ALL THE SAME OR DIFFERENT QUALITIES OF MATHEMATICAL ARGUMENTS?	
Palmér, Hanna	27–34
WHAT HAPPENS WHEN ENTREPRENEURSHIP ENTERS MATHEMATICS LESSONS?	
Papadopoulos, Ioannis; Diamantidis, Dimitris; Kynigos, Chronis	35–42
MEANINGS AROUND ANGLE WITH DIGITAL MEDIA DESIGNED TO SUPPORT CREATIVE MATHEMATICAL THINKING	
Papadopoulos, Ioannis; Kindini, Theonitsa; Tsakalaki, Xanthippi	43–50
USING MOBILE PUZZLES TO DEVELOPE ALGEBRAIC THINKING	
Pelen, Mustafa Serkan; Dinç Artut, Perihan	51–58
AN INVESTIGATION OF MIDDLE SCHOOL STUDENTS’ PROBLEM SOLVING STRATEGIES ON INVERSE PROPORTIONAL PROBLEMS	
Pettersen, Andreas; Nortvedt, Guri A.	59–66
RECOGNISING WHAT MATTERS: IDENTIFYING COMPETENCY DEMANDS IN MATHEMATICAL TASKS	
Pinkernell, Guido	67–74
MAKING SENSE OF DYNAMICALLY LINKED MULTIPLE REPRESENTATIONS OF FUNCTIONS	

Pongsakdi, Nonmanut; Brezovszky, Boglarka; Veermans, Koen; Hannula-Sormunen, Minna; Lehtinen, Erno	75–82
A COMPARATIVE ANALYSIS OF WORD PROBLEMS IN SELECTED THAI AND FINNISH TEXTBOOKS	
Portnov-Neeman, Yelena; Amit, Miriam	83–90
THE EFFECT OF THE EXPLICIT TEACHING METHOD ON LEARNING THE WORKING BACKWARDS STRATEGY	
Potari, Despina; Psycharis, Giorgos; Spiliotopoulou, Vassiliki; Triantafillou, Chrissavgi; Zachariades, Theodossios; Zoupa, Aggeliki	91–98
MATHEMATICS AND SCIENCE TEACHERS' COLLABORATION: SEARCHING FOR COMMON GROUNDS	
Proulx, Jérôme; Simmt, Elaine	99–106
DISTINGUISHING ENACTIVISM FROM CONSTRUCTIVISM: ENGAGING WITH NEW POSSIBILITIES	
Pustelnik, Kolja; Halverscheid, Stefan	107–114
ON THE CONSOLIDATION OF DECLARATIVE MATHEMATICAL KNOWLEDGE AT THE TRANSITION TO TERTIARY EDUCATION	
Rangel, Letícia; Giraldo, Victor; Maculan, Nelson	115–122
CONCEPT STUDY AND TEACHERS' META-KNOWLEDGE: AN EXPERIENCE WITH RATIONAL NUMBERS	
Reinhold, Simone; Wöller, Susanne	123–130
THIRD-GRADERS' BLOCK-BUILDING: HOW DO THEY EXPRESS THEIR KNOWLEDGE OF CUBOIDS AND CUBES?	
Rellensmann, Johanna; Schukajlow, Stanislaw	131–138
ARE MATHEMATICAL PROBLEMS BORING? BOREDOM WHILE SOLVING PROBLEMS WITH AND WITHOUT A CONNECTION TO REALITY FROM STUDENTS' AND PRE-SERVICE TEACHERS' PERSPECTIVES	
Rott, Benjamin; Leuders, Timo	139–146
MATHEMATICAL CRITICAL THINKING: THE CONSTRUCTION AND VALIDATION OF A TEST	
Salle, Alexander; Schumacher, Stefanie; Hattermann, Mathias	147–154
THE PING-PONG-PATTERN – USAGE OF NOTES BY DYADS DURING LEARNING WITH ANNOTATED SCRIPTS	

Scheiner, Thorsten; Pinto, Márcia M. F.	155–162
IMAGES OF ABSTRACTION IN MATHEMATICS EDUCATION: CONTRADICTIONS, CONTROVERSIES, AND CONVERGENCES	
Schindler, Maike; Lilienthal, Achim; Chadalavada, Ravi; Ögren, Magnus	163–170
CREATIVITY IN THE EYE OF THE STUDENT. REFINING INVESTIGATIONS OF MATHEMATICAL CREATIVITY USING EYE-TRACKING GOGGLES.	
Segal, Ruti; Shriki, Atara; Movshovitz-Hadar, Nitsa	171–178
FACILITATING MATHEMATICS TEACHERS' SHARING OF LESSON PLANS	
Shahbari, Juhaina Awawdeh; Tabach, Michal	179–186
DIFFERENT GENERALITY LEVELS IN THE PRODUCT OF A MODELLING ACTIVITY	
Shimada, Isao; Baba, Takuya	187–194
TRANSFORMATION OF STUDENTS' VALUES IN THE PROCESS OF SOLVING SOCIALLY OPEN-ENDED PROBLEMS(2): FOCUSING ON LONG-TERM TRANSFORMATION	
Shinno, Yusuke; Fujita, Taro	195–202
PROSPECTIVE MATHEMATICS TEACHERS' PROOF COMPREHENSION OF MATHEMATICAL INDUCTION: LEVELS AND DIFFICULTIES	
Silverman, Boaz; Even, Ruhama	203–210
PATHS OF JUSTIFICATION IN ISRAELI 7TH GRADE MATHEMATICS TEXTBOOKS	
Skott, Charlotte Krog; Østergaard, Camilla Hellsten	211–218
HOW DOES AN ICT-COMPETENT MATHEMATICS TEACHER BENEFIT FROM AN ICT-INTEGRATIVE PROJECT?	
Sommerhoff, Daniel; Ufer, Stefan; Kollar, Ingo	219–226
PROOF VALIDATION ASPECTS AND COGNITIVE STUDENT PREREQUISITES IN UNDERGRADUATE MATHEMATICS	
Staats, Susan	227–234
POETIC STRUCTURES AS RESOURCES FOR PROBLEM- SOLVING	

Stouraitis, Konstantinos	235–242
DECISION MAKING IN THE CONTEXT OF ENACTING A NEW CURRICULUM: AN ACTIVITY THEORETICAL PERSPECTIVE	
Strachota, Susanne M.; Fonger, Nicole L.; Stephens, Ana C.; Blanton, Maria L.; Knuth, Eric J.; Murphy Gardiner, Angela	243–250
UNDERSTANDING VARIATION IN ELEMENTARY STUDENTS' FUNCTIONAL THINKING	
Sumpter, Lovisa; Sumpter, David	251–258
HOW LONG WILL IT TAKE TO HAVE A 60/40 BALANCE IN MATHEMATICS PHD EDUCATION IN SWEDEN?	
Tabach, Michal; Hershkowitz, Rina; Azmon, Shirly; Rasmussen, Chris; Dreyfus, Tommy	259–266
TRACES OF CLASSROOM DISCOURSE IN A POSTTEST	
Takeuchi, Miwa; Towers, Jo; Martin, Lyndon	267–274
IMAGES OF MATHEMATICS LEARNING REVEALED THROUGH STUDENTS' EXPERIENCES OF COLLABORATION	
Tjoe, Hartono	275–282
WHEN IS A PROBLEM REALLY SOLVED? DIFFERENCES IN THE PURSUIT OF MATHEMATICAL AESTHETICS	
Triantafillou, Chrissavgi; Bakogianni, Dionysia; Kosyvas, Georgios	283–290
TENSIONS IN STUDENTS' GROUP WORK ON MODELLING ACTIVITIES	
Uegatani, Yusuke; Koyama, Masataka	291–298
A NEW FRAMEWORK BASED ON THE METHODOLOGY OF SCIENTIFIC RESEARCH PROGRAMS FOR DESCRIBING THE QUALITY OF MATHEMATICAL ACTIVITIES	
Ulusoy, Fadime	299–306
THE ROLE OF LEARNERS' EXAMPLE SPACES IN EXAMPLE GENERATION AND DETERMINATION OF TWO PARALLEL AND PERPENDICULAR LINE SEGMENTS	
Uziel, Odelya; Amit, Miriam	307–314
COGNITIVE AND AFFECTIVE CHARACTERISTICS OF YOUNG SOLVERS PARTICIPATING IN 'KIDUMATICA FOR YOUTH'	

Van Hoof, Jo; Verschaffel, Lieven; Ghesquière, Pol; Van Dooren, Wim	315–322
THE NATURAL NUMBER BIAS AND ITS ROLE IN RATIONAL NUMBER UNDERSTANDING IN CHILDREN WITH DYSCALCULIA: DELAY OR DEFICIT?	
Van Zoest, Laura R.; Stockero, Shari L.; Leatham, Keith R.; Peterson, Blake E.	323–330
THEORIZING THE MATHEMATICAL POINT OF BUILDING ON STUDENT MATHEMATICAL THINKING	
Vázquez Monter, Nathalie	331–338
INCORPORATING MOBILE TECHNOLOGIES INTO THE PRE- CALCULUS CLASSROOM: A SHIFT FROM TI GRAPHIC CALCULATORS TO PERSONAL MOBILE DEVICES	
Vermeulen, Cornelis	339–346
DEVELOPING ALGEBRAIC THINKING: THE CASE OF SOUTH AFRICAN GRADE 4 TEXTBOOKS.	
Vlassis, Jöelle; Poncelet, Débora	347–354
PRE-SERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS EDUCATION FOR 3-6-YEAR-OLD CHILDREN	
Waisman, Ilana	355–362
ENLISTING PHYSICS IN THE SERVICE OF MATHEMATICS: FOCUSSING ON HIGH SCHOOL TEACHERS	
Walshaw, Margaret	363–370
REFLECTIVE PRACTICE AND TEACHER IDENTITY: A PSYCHOANALYTIC VIEW	
Wang, Ting-Ying; Hsieh, Feng-Jui	371–378
WHAT TEACHERS SHOULD DO TO PROMOTE AFFECTIVE ENGAGEMENT WITH MATHEMATICS—FROM THE PERSPECTIVE OF ELEMENTARY STUDENTS	
Wasserman, Nicholas H.	379–386
NONLOCAL MATHEMATICAL KNOWLEDGE FOR TEACHING	
Wilkie, Karina J.	387–394
EXPLORING MIDDLE SCHOOL GIRLS' AND BOYS' ASPIRATIONS FOR THEIR MATHEMATICS LEARNING	

Xenofontos, Constantinos; Kyriakou, Artemis	395–402
PROSPECTIVE ELEMENTARY TEACHERS' TALK DURING COLLABORATIVE PROBLEM SOLVING	
Zeljić, Marijana; Đokić, Olivera; Dabić, Milana	403–410
TEACHERS' BELIEFS TOWARDS THE VARIOUS REPRESENTATIONS IN MATHEMATICS INSTRUCTION	
Index of Authors	413–414

IMAGES OF ABSTRACTION IN MATHEMATICS EDUCATION: CONTRADICTIONS, CONTROVERSIES, AND CONVERGENCES

Thorsten Scheiner¹ & Márcia M. F. Pinto²

¹University of Hamburg, Germany; ²Federal University of Rio de Janeiro, Brazil

In this paper we offer a critical reflection of the mathematics education literature on abstraction. We explore several explicit or implicit basic orientations, or what we call images, about abstraction in knowing and learning mathematics. Our reflection is intended to provide readers with an organized way to discern the contradictions, controversies, and convergences concerning the many images of abstraction. Given the complexity and multidimensionality of the notion of abstraction, we argue that seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. We suggest considering abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas by acknowledging knowledge as a complex system.

INTRODUCTION

Several scholars in the psychology of mathematics education have recognized abstraction to be one of the key traits in mathematics learning and thinking (e.g., Boero et al., 2002). The literature acknowledges a variety of forms of abstraction (Dreyfus, 2014) that take place at different levels of mathematical learning (Mitchelmore & White, 2012) or in different worlds of mathematics (Tall, 2013), and underlie different ways of constructing mathematical concepts compatible with various sense-making strategies (Scheiner, 2016). While the complexity and multi-dimensionality of abstraction is widely documented (e.g., Boero et al., 2002; Dreyfus, 1991), the literature lacks a discourse on – conflicting, controversial, and converging – images of abstraction in mathematics education.

In this article, we offer a reflection on the literature on abstraction in mathematics learning that is somewhat at variance with other reflections and overviews. We explicitly focus on what key writings in this realm assert, assume, and imply about the nature of abstraction in mathematics education. Much of the literature is concerned with a discussion about the multiplicity and diversity of approaches and with frameworks of abstraction; however, what is missing is an articulation of basic orientations or images of abstraction. Our reflection is intended to provide readers with an organized way to discern the controversies, contradictions, and convergences of the many images of abstraction that are explicit or implicit in the literature.

The three following sections consider each of the above facets (contradictions, controversies, and convergences), and relate our reflections on the literature regarding abstraction in mathematics education. We approach each of them by presenting issues that in our view are central to the debate. We conclude with some remarks on viewing

knowledge as a complex dynamic system that acknowledges abstraction in terms of levels of complexity and increases in context-sensitivity.

SOME CONTRADICTING IMAGES OF ABSTRACTION

We take the following description of abstraction by Fuchs et al. (2003) as a starting point for discussing the main contradicting images of abstraction still present in the literature:

“To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category [...]. These abstractions [...] avoid contextual specificity so they can be applied to other instances or across situations.” (Fuchs et al., 2003, p. 294)

The contradicting image of abstraction as generalization

The description of abstraction given by Fuchs et al. (2003) focuses on the generality, or, rather, on the generic quality of a concept. Here abstraction is identified with generalization. Generalization of a concept implies taking away a certain number of attributes from a specific concept. For example, taking away the attribute ‘to have orthogonal sides’ from the concept of rectangle leads to the concept of parallelogram. This operation implies an extension of the scope of the concept and forms a more general concept.

Abstraction, in contrast, does not mean taking away but *extracting* and *attributing* certain meaningful components. In considering forms of abstraction on the background of students’ sense-making, Scheiner (2016) argued that ‘abstractions from actions’ approaches (e.g., reflective abstraction) are compatible with students’ sense-making strategy of ‘extracting meaning’ and ‘abstractions from objects’ approaches (e.g., structural abstraction) are compatible with students’ sense-making strategy of ‘giving meaning’ – two prototypical sense-making strategies identified by Pinto (1998). From this perspective, in attributing meaningful components, one’s concept image becomes richer in content.

Thus, the image of abstraction as generalization seems inadequate when knowledge is considered as construction. The image of abstraction as generalization is elusive about abstraction as a constructive process and overlooks abstraction that takes account of an individual’s cognitive development.

The contradicting image of abstraction as decontextualization

The above quoted description of abstraction by Fuchs et al. (2003) implies that abstraction is concerned with a certain degree of decontextualization. This is not surprising, given the confusion of abstraction with generalization as “generalization and decontextualization [often] act as two sides of the same coin” (Ferrari, 2003, p. 1226). Fuchs et al. (2003) suggested getting away from contextual specificities so that “abstractions [...] can be applied to other instances or across situations” (p. 294). Furthermore, the meaning *abstract-general* of the term ‘abstract’ (Mitchelmore &

White, 1995), refers to ideas which are general to a wide variety of contexts, and this may cause such confusions.

The consideration of abstraction as decontextualization contradicts the recent advances in understanding knowledge as situated and context sensitive (e.g., Brown, Collins, & Duguid, 1989; Cobb & Bowers, 1999). Several scholars in mathematics education have argued against the decontextualization view of abstraction. For example, Noss and Hoyles' (1996) *situated abstraction* approach and Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context* framework have foregrounded the significance of context for abstraction processes in mathematics learning and thinking. These contributions go beyond purely cognitive approaches and frameworks of abstraction in mathematics education and take account of the situated nature and context-sensitivity of knowledge, as articulated by the situated cognition (or situated learning) paradigm. van Oers (1998) focussed on this aspect in arguing that abstraction is a kind of *recontextualization* rather than a *decontextualization*. From his perspective, removing context will impoverish a concept rather than enrich it. Scheiner and Pinto (2014) presented a case study in which a student integrated diverse elements of representing the limit concept of a sequence into a single representation that the student used generically to construct and reconstruct the limit concept in multiple contexts. Their analysis indicated that the representation (that the student constructed) supported his actions through its complex sensitivity to the contextual differences he encountered.

Thus, from our point of view, we acknowledge abstraction as a process of increasing context-sensitivity rather than considering abstraction as simply decontextualization.

SOME CONTROVERSIAL IMAGES OF ABSTRACTION

The controversial image of abstraction on structures: similarity or diversity?

Theoretical research in learning mathematics has long moved beyond categorization or classification, that is, beyond collecting together objects on the basis of similarities of their superficial characteristics. As diSessa and Sherin (1998) reminded us, though abstraction as derived from the recognition of commonalities of properties works well for 'category-like concepts', empirical approaches limited to the perceptual characteristics of objects do not provide fertile insights into cognitive processes underlying concept construction in mathematics. Skemp's (1986) idea of abstraction, that is, of studying the underlying structure rather than superficial characteristics moved the field in new directions. Further, Mitchelmore and White (2000), in drawing on Skemp's conception of abstraction, developed an empirical abstraction approach for learning elementary mathematics.

Though the literature portrays a mutual understanding that abstraction in mathematics is concerned with the underlying (rather than the superficial) structures of a concept, there is a controversy as to whether abstraction means the consideration of similarities of structures or of their diversity. While Skemp (1986) focused on similarities in structures, Vygotsky (1934/1987) considered the formation of scientific concepts along differences.

A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole. [...] Such a concept [...] traces the interconnection of particular objects within the whole, within the system in its formation. (Vygotsky, 1934/1987, p. 255)

Scheiner (2016) proposed a framework for structural abstraction, a kind of abstraction, already introduced by Tall (2013), that takes account of abstraction as a process of complementarizing meaningful components. From this perspective, the meaning of mathematical concepts is constructed by complementarizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations.

Thus, it is still debated whether the meaning of a mathematical concept relies on the commonality of elements or on the interrelatedness of diverse elements – or, to put it in other words, whether the core of abstraction is similarity or complementarity.

The controversial image of abstraction as the ascending of abstractness or complexity

Scholars seem to agree in distinguishing between concrete and abstract objects, yet not between concrete and abstract concepts since every concept is an abstraction. In fact, scholars differ with regard to their understanding of the notions of ‘concrete’ and ‘abstract’. According to Skemp (1986), the initial forms of cognition are perceptions of concrete objects; the abstractions from concrete objects are called percepts. These percepts are considered primary concepts and serve as building blocks for secondary concepts; the latter are concepts that do not have to correspond to any concrete object. Taking this perspective, it is not surprising that concreteness and abstractness are often considered as properties of an object. In contrast, Wilensky (1991) considered concreteness and abstractness rather as properties of an individual’s relatedness to an object in the sense of the richness of an individual’s re-presentations, interactions, and connections with the object. This view leads to allowing objects not mediated by the senses, objects which are usually considered abstract (such as mathematical objects) to be concrete; as long as that the individual has multiple modes of interaction and connection with them and a sufficiently rich collection of representations to denote them.

Skemp viewed abstraction as a movement from the concrete to the abstract, while, according to Wilensky, individuals begin their understanding of scientific mathematical concepts with the abstract. This ascending from the abstract to the concrete is the main principle in Davydov’s (1972/1990) theory and has been taken as a reference point for the development of other frameworks of abstraction (e.g., Hershkowitz, Dreyfus, & Schwarz, 2001; Scheiner, 2016).

On the other hand, Noss and Hoyles (1996) adopted a situated cognition perspective to investigate mathematical activities within computational environments. These environments are specially built to provide learners an opportunity for new intellectual connections. The authors’ concern is “to develop a conscious appreciation of

mathematical abstraction as a process which builds upon layers of intuitions and meanings” (Noss & Hoyles, 1996, p. 105).

Thus, in taking the understanding of the concrete and the abstract as properties of objects, scholars could consider abstraction as levels of abstractness; while, in taking the understanding of concreteness and abstractness as properties of an individual’s view of objects, scholars could view abstraction as levels of complexity, as Scheiner and Pinto’s (2014) recent contribution indicated.

SOME CONVERGING IMAGES OF ABSTRACTION

Piaget (1977/2001) made a distinction between cognitive approaches to abstraction: dichotomizing ‘abstraction from actions’ and ‘abstraction from objects’. Research in mathematics education has mostly considered the first of these approaches to abstraction. In referring to the latter, Piaget (1977/2001) limited his attention to empirical abstraction, that is, to drawing out common features of objects, “recording the most obvious information from objects” (p. 319). Supported by Skemp’s view on abstraction, Mitchelmore and White (2000), and later Scheiner and Pinto (2014), considered objects as starting points for abstraction processes, and, in doing so, took account of ‘abstraction from objects’. Scheiner (2016) blended the abstraction from actions and the abstraction from objects frameworks to provide an account for a dialectic between reflective and structural abstraction. In the following, we provide convergent images of these various notions of abstraction, as we see them.

The converging image of abstraction as a process of knowledge compression

Here we understand compression of knowledge as “taking complicated phenomena, focusing on essential aspects of interest to conceive of them as whole to make them available as an entity to think about” (Gray & Tall, 2007, p. 24). Or, to put it in Thurston’s (1990) words, knowledge is compressed if “you can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process” (p. 847).

Dubinsky and his colleagues’ (Dubinsky, 1991; Cottrill et al., 1996) APOS framework, which seems to refer mostly to ‘abstraction from actions’, proposed the notion of encapsulation of processes into an object through what Piaget called reflective abstraction. The single encapsulated object may be understood as a compression in a sense that encapsulation results in an entity to think about. The same holds for Sfard and Linchevski’s (1994) framework of reification, a process that results in a structural conception of an object. In the same strand, Gray and Tall (1994) considered some mathematical symbols as an amalgam of processes and related objects; thus, compressing knowledge into a symbol which is conveniently understood as a process to compute or manipulate, or as a concept to think about. They proposed that “the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking” (Gray & Tall, 2007, p. 14).

Researchers working within the ‘abstraction from objects’ strand (Mitchelmore & White, 2000; Scheiner & Pinto, 2014) are guided by the assumption that learners acquire mathematical concepts initially based on their backgrounds of existing domain-specific conceptual knowledge – considering abstraction as the progressive integration of previous concept images and/or the insertion of a new discourse alongside existing mathematical experiences. For instance, the cognitive function of structural abstraction is to provide an assembly of such various experiences into more complex and compressed knowledge structures (Scheiner & Pinto, 2014).

Thus, both ‘abstraction from actions’ and ‘abstraction from objects’ approaches seem to share the image of abstraction as a process of knowledge compression.

The converging image of abstraction as a complex dynamic constructive process

One may argue that researchers who see abstraction as decontextualization propose the result of an abstraction process as a stable stage. Once decontextualized, the product of an abstraction – the concept – appears as standing still. An understanding of the entire process as a recontextualization considers abstraction to be a dynamic constructive process, which could evolve in a movement through levels of complexity. In fact, concepts can be continuously revised and enriched while placed in new contexts. This seems to agree with the understanding of Noss and Hoyles (1996) and of Hershkowitz, Schwarz and Dreyfus (2001). In the case of Scheiner and Pinto (2014), the underlying cognitive processes support a specific use of the concept image while building mathematical knowledge. Models of partial constructions are gradually built through these processes and are used as generic representations. In other words, a model of an evolving concept is built and used for generating meaningful components as needed, while inducing a restructuring of one’s knowledge system. From this perspective, an individual’s restructuring of the knowledge system aims for stability of the knowledge pieces and structures. Such dynamic constructive processes emphasize a gradually developing process of knowledge construction.

Thus, rather than considering knowledge as an abstract, stable system, we consider knowledge as a complex dynamic system of various types of knowledge elements and structures.

FINAL REMARKS

This brief discussion underlines the many images of abstraction in mathematics learning and thinking. If abstraction is regarded from the viewpoint of knowledge as a static system, then abstraction refers to meanings that are ‘abstracted’ from situations or events. By taking this view, abstraction is considered as a highly hierarchized process, whereby abstractions of higher order are built upon abstractions of lower order. However, if we consider knowledge as a complex system, it is possible to acknowledge abstraction in terms of levels of complexity and increases in context-sensitivity. In viewing knowledge as a complex dynamic system rather than a static system, seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. The central assertion of all approaches and frameworks

should be to consider abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas.

Acknowledgments

We want to thank Annie Selden for her thoughtful comments and suggestions given throughout the development of this paper.

References

- Boero, P., Dreyfus, T., Gravemeijer, K., Gray, E., Hershkowitz, R., Schwarz, B., Sierpiska, A., & Tall, D. O. (2002). Abstraction: Theories about the emergence of knowledge structures. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 113-138). Norwich: PME.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-42.
- Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher*, 28(2), 4-15.
- Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thornas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, 15, 167-192.
- Davydov, V. V. (1972/1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2) (translated by J. Teller). Reston, VA: NCTM.
- diSessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, 20(10), 1155-1191.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.
- Dreyfus, T. (2014). Abstraction in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 5-8). New York: Springer.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer.
- Ferrari, P. L. (2003). Abstraction in mathematics. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 358, 1225-1230.
- Fuchs, L. S., Fuchs, D., Prentice, K., Burch, M., Hamlett, C. L., Owen, R., et al. (2003). Explicitly teaching transfer: Effects on third-grade students' mathematical problem solving. *Journal of Educational Psychology*, 95(2), 293-305.
- Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 26(2), 115-141.
- Gray, E. M., & Tall, D. O. (2007). Abstraction as a natural process of mental compression. *Mathematics Education Research Journal*, 19(2), 23-40

- Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32, 195-222.
- Mitchelmore, M. C., & White, P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal*, 7(1), 50-68.
- Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, 41(3), 209-238.
- Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer.
- Piaget, J. [and collaborators] (1977/2001). *Studies in reflecting abstraction* (Recherches sur l'abstraction réfléchissante) (translated by R. Campbell). Philadelphia: Psychology Press.
- Pinto, M. M. F. (1998). *Students' understanding of real analysis*. Coventry, UK: University of Warwick.
- Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, 91(2), 165-183.
- Scheiner, T., & Pinto, M. M. F. (2014). Cognitive processes underlying mathematical concept construction: The missing process of structural abstraction. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 5, pp. 105-112). Vancouver, Canada: PME.
- Sfard, A., & Lichevski, L. (1994). The gains and the pitfalls of reification: the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.
- Skemp, R. R. (1986). *The psychology of learning mathematics*. (Second edition, first published 1971). London: Penguin Group.
- Tall, D. O. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics*. Cambridge, UK: Cambridge University Press.
- van Oers, B. (1998). From context to contextualizing. *Learning and Instruction*, 8(6), 473-488.
- Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 193-204). Norwood, NJ: Ablex Publishing Corporation.