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IMAGES OF ABSTRACTION IN MATHEMATICS EDUCATION: CONTRADICTIONS, CONTROVERSIES, AND CONVERGENCES

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In this paper we offer a critical reflection of the mathematics education literature on abstraction. We explore several explicit or implicit basic orientations, or what we call images, about abstraction in knowing and learning mathematics. Our reflection is intended to provide readers with an organized way to discern the contradictions, controversies, and convergences concerning the many images of abstraction. Given the complexity and multidimensionality of the notion of abstraction, we argue that seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. We suggest considering abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas by acknowledging knowledge as a complex system.

INTRODUCTION

Several scholars in the psychology of mathematics education have recognized abstraction to be one of the key traits in mathematics learning and thinking (e.g., Boero et al., 2002). The literature acknowledges a variety of forms of abstraction (Dreyfus, 2014) that take place at different levels of mathematical learning (Mitchelmore & White, 2012) or in different worlds of mathematics (Tall, 2013), and underlie different ways of constructing mathematical concepts compatible with various sense-making strategies (Scheiner, 2016). While the complexity and multi-dimensionality of abstraction is widely documented (e.g., Boero et al., 2002; Dreyfus, 1991), the literature lacks a discourse on – conflicting, controversial, and converging – images of abstraction in mathematics education.

In this article, we offer a reflection on the literature on abstraction in mathematics learning that is somewhat at variance with other reflections and overviews. We explicitly focus on what key writings in this realm assert, assume, and imply about the nature of abstraction in mathematics education. Much of the literature is concerned with a discussion about the multiplicity and diversity of approaches and with frameworks of abstraction; however, what is missing is an articulation of basic orientations or images of abstraction. Our reflection is intended to provide readers with an organized way to discern the controversies, contradictions, and convergences of the many images of abstraction that are explicit or implicit in the literature.

The three following sections consider each of the above facets (contradictions, controversies, and convergences), and relate our reflections on the literature regarding abstraction in mathematics education. We approach each of them by presenting issues that in our view are central to the debate. We conclude with some remarks on viewing

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knowledge as a complex dynamic system that acknowledges abstraction in terms of levels of complexity and increases in context-sensitivity.

SOME CONTRADICTING IMAGES OF ABSTRACTION

We take the following description of abstraction by Fuchs et al. (2003) as a starting point for discussing the main contradicting images of abstraction still present in the literature:

> "To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category […]. These abstractions […] avoid contextual specificity so they can be applied to other instances or across situations." (Fuchs et al., 2003, p. 294)

The contradicting image of abstraction as generalization

The description of abstraction given by Fuchs et al. (2003) focuses on the generality, or, rather, on the generic quality of a concept. Here abstraction is identified with generalization. Generalization of a concept implies taking away a certain number of attributes from a specific concept. For example, taking away the attribute 'to have orthogonal sides' from the concept of rectangle leads to the concept of parallelogram. This operation implies an extension of the scope of the concept and forms a more general concept.

Abstraction, in contrast, does not mean taking away but *extracting* and *attributing* certain meaningful components. In considering forms of abstraction on the background of students' sense-making, Scheiner (2016) argued that 'abstractions from actions' approaches (e.g., reflective abstraction) are compatible with students' sense-making strategy of 'extracting meaning' and 'abstractions from objects' approaches (e.g., structural abstraction) are compatible with students' sense-making strategy of 'giving meaning' – two prototypical sense-making strategies identified by Pinto (1998). From this perspective, in attributing meaningful components, one's concept image becomes richer in content.

Thus, the image of abstraction as generalization seems inadequate when knowledge is considered as construction. The image of abstraction as generalization is elusive about abstraction as a constructive process and overlooks abstraction that takes account of an individual's cognitive development.

The contradicting image of abstraction as decontextualization

The above quoted description of abstraction by Fuchs et al. (2003) implies that abstraction is concerned with a certain degree of decontextualization. This is not surprising, given the confusion of abstraction with generalization as "generalization and decontextualization [often] act as two sides of the same coin" (Ferrari, 2003, p. 1226). Fuchs et al. (2003) suggested getting away from contextual specificities so that "abstractions […] can be applied to other instances or across situations" (p. 294). Furthermore, the meaning *abstract-general* of the term 'abstract' (Mitchelmore &

White, 1995), refers to ideas which are general to a wide variety of contexts, and this may cause such confusions.

The consideration of abstraction as decontextualization contradicts the recent advances in understanding knowledge as situated and context sensitive (e.g., Brown, Collins, & Duguid, 1989; Cobb & Bowers, 1999). Several scholars in mathematics education have argued against the decontextualization view of abstraction. For example, Noss and Hoyles' (1996) *situated abstraction* approach and Hershkowitz, Schwarz, and Dreyfus' (2001) *abstraction in context* framework have foregrounded the significance of context for abstraction processes in mathematics learning and thinking. These contributions go beyond purely cognitive approaches and frameworks of abstraction in mathematics education and take account of the situated nature and context-sensitivity of knowledge, as articulated by the situated cognition (or situated learning) paradigm. van Oers (1998) focussed on this aspect in arguing that abstraction is a kind of *re*contextualization rather than a *de*contextualization. From his perspective, removing context will impoverish a concept rather than enrich it. Scheiner and Pinto (2014) presented a case study in which a student integrated diverse elements of representing the limit concept of a sequence into a single representation that the student used generically to construct and reconstruct the limit concept in multiple contexts. Their analysis indicated that the representation (that the student constructed) supported his actions through its complex sensitivity to the contextual differences he encountered.

Thus, from our point of view, we acknowledge abstraction as a process of increasing context-sensitivity rather than considering abstraction as simply decontextualization.

SOME CONTROVERSIAL IMAGES OF ABSTRACTION

The controversial image of abstraction on structures: similarity or diversity?

Theoretical research in learning mathematics has long moved beyond categorization or classification, that is, beyond collecting together objects on the basis of similarities of their superficial characteristics. As diSessa and Sherin (1998) reminded us, though abstraction as derived from the recognition of commonalities of properties works well for 'category-like concepts', empirical approaches limited to the perceptual characteristics of objects do not provide fertile insights into cognitive processes underlying concept construction in mathematics. Skemp's (1986) idea of abstraction, that is, of studying the underlying structure rather than superficial characteristics moved the field in new directions. Further, Mitchelmore and White (2000), in drawing on Skemp's conception of abstraction, developed an empirical abstraction approach for learning elementary mathematics.

Though the literature portrays a mutual understanding that abstraction in mathematics is concerned with the underlying (rather than the superficial) structures of a concept, there is a controversy as to whether abstraction means the consideration of similarities of structures or of their diversity. While Skemp (1986) focused on similarities in structures, Vygotsky (1934/1987) considered the formation of scientific concepts along differences.

> A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole. [...] Such a concept [...] traces the interconnection of particular objects within the whole, within the system in its formation. (Vygotsky, 1934/1987, p. 255)

Scheiner (2016) proposed a framework for structural abstraction, a kind of abstraction, already introduced by Tall (2013), that takes account of abstraction as a process of complementarizing meaningful components. From this perspective, the meaning of mathematical concepts is constructed by complementarizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations.

Thus, it is still debated whether the meaning of a mathematical concept relies on the commonality of elements or on the interrelatedness of diverse elements – or, to put it in other words, whether the core of abstraction is similarity or complementarity.

The controversial image of abstraction as the ascending of abstractness or complexity

Scholars seem to agree in distinguishing between concrete and abstract objects, yet not between concrete and abstract concepts since every concept is an abstraction. In fact, scholars differ with regard to their understanding of the notions of 'concrete' and 'abstract'. According to Skemp (1986), the initial forms of cognition are perceptions of concrete objects; the abstractions from concrete objects are called percepts. These percepts are considered primary concepts and serve as building blocks for secondary concepts; the latter are concepts that do not have to correspond to any concrete object. Taking this perspective, it is not surprising that concreteness and abstractness are often considered as properties of an object. In contrast, Wilensky (1991) considered concreteness and abstractness rather as properties of an individual's relatedness to an object in the sense of the richness of an individual's re-presentations, interactions, and connections with the object. This view leads to allowing objects not mediated by the senses, objects which are usually considered abstract (such as mathematical objects) to be concrete; as long as that the individual has multiple modes of interaction and connection with them and a sufficiently rich collection of representations to denote them.

Skemp viewed abstraction as a movement from the concrete to the abstract, while, according to Wilensky, individuals begin their understanding of scientific mathematical concepts with the abstract. This ascending from the abstract to the concrete is the main principle in Davydov's (1972/1990) theory and has been taken as a reference point for the development of other frameworks of abstraction (e.g., Hershkowitz, Dreyfus, & Schwarz, 2001; Scheiner, 2016).

On the other hand, Noss and Hoyles (1996) adopted a situated cognition perspective to investigate mathematical activities within computational environments. These environments are specially built to provide learners an opportunity for new intellectual connections. The authors' concern is "to develop a conscious appreciation of mathematical abstraction as a process which builds build upon layers of intuitions and meanings" (Noss & Hoyles, 1996, p. 105).

Thus, in taking the understanding of the concrete and the abstract as properties of objects, scholars could consider abstraction as levels of abstractness; while, in taking the understanding of concreteness and abstractness as properties of an individual's view of objects, scholars could view abstraction as levels of complexity, as Scheiner and Pinto's (2014) recent contribution indicated.

SOME CONVERGING IMAGES OF ABSTRACTION

Piaget (1977/2001) made a distinction between cognitive approaches to abstraction: dichotomizing 'abstraction from actions' and 'abstraction from objects'. Research in mathematics education has mostly considered the first of these approaches to abstraction. In referring to the latter, Piaget (1977/2001) limited his attention to empirical abstraction, that is, to drawing out common features of objects, "recording the most obvious information from objects" (p. 319). Supported by Skemp's view on abstraction, Mitchelmore and White (2000), and later Scheiner and Pinto (2014), considered objects as starting points for abstraction processes, and, in doing so, took account of 'abstraction from objects'. Scheiner (2016) blended the abstraction from actions and the abstraction from objects frameworks to provide an account for a dialectic between reflective and structural abstraction. In the following, we provide convergent images of these various notions of abstraction, as we see them.

The converging image of abstraction as a process of knowledge compression

Here we understand compression of knowledge as "taking complicated phenomena, focusing on essential aspects of interest to conceive of them as whole to make them available as an entity to think about" (Gray & Tall, 2007, p. 24). Or, to put it in Thurston's (1990) words, knowledge is compressed if "you can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process" (p. 847).

Dubinsky and his colleagues' (Dubinsky, 1991; Cottrill et al., 1996) APOS framework, which seems to refer mostly to 'abstraction from actions', proposed the notion of encapsulation of processes into an object through what Piaget called reflective abstraction. The single encapsulated object may be understood as a compression in a sense that encapsulation results in an entity to think about. The same holds for Sfard and Linchevski's (1994) framework of reification, a process that results in a structural conception of an object. In the same strand, Gray and Tall (1994) considered some mathematical symbols as an amalgam of processes and related objects; thus, compressing knowledge into a symbol which is conveniently understood as a process to compute or manipulate, or as a concept to think about. They proposed that "the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking" (Gray & Tall, 2007, p. 14).

Researchers working within the 'abstraction from objects' strand (Mitchelmore & White, 2000; Scheiner & Pinto, 2014) are guided by the assumption that learners acquire mathematical concepts initially based on their backgrounds of existing domain-specific conceptual knowledge – considering abstraction as the progressive integration of previous concept images and/or the insertion of a new discourse alongside existing mathematical experiences. For instance, the cognitive function of structural abstraction is to provide an assembly of such various experiences into more complex and compressed knowledge structures (Scheiner & Pinto, 2014).

Thus, both 'abstraction from actions' and 'abstraction from objects' approaches seem to share the image of abstraction as a process of knowledge compression.

The converging image of abstraction as a complex dynamic constructive process

One may argue that researchers who see abstraction as decontextualization propose the result of an abstraction process as a stable stage. Once decontextualized, the product of an abstraction – the concept – appears as standing still. An understanding of the entire process as a recontextualization considers abstraction to be a dynamic constructive process, which could evolve in a movement through levels of complexity. In fact, concepts can be continuously revised and enriched while placed in new contexts. This seems to agree with the understanding of Noss and Hoyles (1996) and of Hershkowitz, Schwarz and Dreyfus (2001). In the case of Scheiner and Pinto (2014), the underlying cognitive processes support a specific use of the concept image while building mathematical knowledge. Models of partial constructions are gradually built through these processes and are used as generic representations. In other words, a model of an evolving concept is built and used for generating meaningful components as needed, while inducing a restructuring of one's knowledge system. From this perspective, an individual's restructuring of the knowledge system aims for stability of the knowledge pieces and structures. Such dynamic constructive processes emphasize a gradually developing process of knowledge construction.

Thus, rather than considering knowledge as an abstract, stable system, we consider knowledge as a complex dynamic system of various types of knowledge elements and structures.

FINAL REMARKS

This brief discussion underlines the many images of abstraction in mathematics learning and thinking. If abstraction is regarded from the viewpoint of knowledge as a static system, then abstraction refers to meanings that are 'abstracted' from situations or events. By taking this view, abstraction is considered as a highly hierarchized process, whereby abstractions of higher order are built upon abstractions of lower order. However, if we consider knowledge as a complex system, it is possible to acknowledge abstraction in terms of levels of complexity and increases in context-sensitivity. In viewing knowledge as a complex dynamic system rather than a static system, seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. The central assertion of all approaches and frameworks should be to consider abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas.

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