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ORAL COMMUNICATIONS

POSTER PRESENTATIONS

THEORETICAL ADVANCES IN MATHEMATICAL COGNITION

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This paper articulates and explicates theoretical perspectives that emerged in accounting for the complex dynamic processes involved when individuals ascribe meaning to the mathematical objects of their thinking. Here the focus is on the following processes that are convoluted in the complex dynamics in mathematical concept formation: contextualizing, complementizing, and complexifying. The paper elaborates these three processes in detail, recognizing their epistemological, conceptual, and cognitive significance in mathematical knowing and learning.

INTRODUCTION

Theoretical advancement is key to driving progress in mathematics education research and practice, and the deep understanding it can foster is essential when confronting fundamental problems. However, as diSessa (1991) asserted, in the learning sciences "theory is in a poor state" (p. 221), and the mathematics education community has "not reached deep theoretical understanding of knowledge or the learning process" (p. 221). For diSessa (1991), this is problematic particularly as "intuitive frames are not powerful enough to constitute theories of the mind in general and learning in particular" (p. 225). Reaching deep theoretical understanding of knowing and learning mathematics is challenging not only due to the complexity of phenomena under consideration but also because these phenomena are studied from a diversity of viewpoints both socially and culturally situated (Sierpinska & Kilpatrick, 1998) and relying on different philosophies and paradigms (Cobb, 2007).

Over the past two decades, various theoretical frameworks have arisen to account for cognitive development in mathematical knowing and learning. Here the focus is explicitly on local theories of knowing and learning in mathematics education to explain a specific set of phenomena, instead of global theories that are often tools to produce knowledge of or about mathematics education. Such local theories "are constructions in a state of flux" (Bikner-Ahsbahs & Prediger, 2010, p. 488) that shape, and are shaped by, research practices. This paper outlines some of the theoretical advances gained in our recent research that has been dedicated to better accounting for the complexity of mathematical knowing and learning on a fine-grained level.

Over the past five years, we explored critical processes in mathematical cognition and searched for dialogical possibilities to both move the discussion beyond simple comparison and offer new insights into complex phenomena in mathematical knowing and learning. In Scheiner (2016), two seemingly opposing forms of abstraction (i.e., abstraction from actions and abstraction from objects; Piaget, 1977/2001) and sense-

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making strategies when learning formal mathematics (i.e., extracting meaning and giving meaning; Pinto, 1998, 2018) were put in dialogue. This dialogue contributed to reconsidering the notion of abstraction – as ascribing meaning to the objects of an individual's thinking from a perspective an individual has taken rather than as recognizing a previously unnoticed meaning of a concept (for a discussion of different images of abstraction, see Scheiner & Pinto, 2016). Within this reinterpretation, meaning is construed not as an inherent quality of objects to be extracted, but something that is attributed to objects of one's thinking. To this end, Scheiner's (2016) theoretical discussion acknowledged three processes as central to mathematical concept formation that are the substance of this paper, namely contextualizing, complementizing, and complexifying.

This paper reports theoretical perspectives and insights gained over the past few years that advance our understanding of contextualizing, complementizing, and complexifying, particularly concerning their epistemological, conceptual, and cognitive significance in mathematical knowing and learning. These new perspectives and insights inform research on mathematical cognition and enable one to see not only new phenomena in mathematical concept formation, but to think about these phenomena differently.

THEORETICAL ORIENTATIONS AND ORIENTING ASSERTIONS

The theoretical perspectives put forth here emerged as elaborations of a diversity of points of view on mathematical knowing and learning, organized around critical insights provided by the German mathematician and philosopher Gottlob F. L. Frege (1848-1925). Here we cultivate these theoretical insights as means of advancing our understanding of at least two critical issues involved in mathematical cognition. First, we share Frege's (1892a) assertion that a mathematical concept is not directly accessible through the concept itself but only through objects that act as proxies for it. Second, mathematical objects (unlike objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the object), but only via some 'mode of presentation' (Frege, 1892b) – that is, objects need to be expressed by using signs or other semiotic means such as a gestures, pictures, or linguistic expression (Radford, 2002). The 'mode of presentation' of an object is to be distinguished from the object that is represented, as individuals often confuse a sense$_F$ ('Sinn') of an expression (or representation) with the reference$_F$ ('Bedeutung') of an expression (or representation) (the subscript F indicates that these terms refer to Frege, 1892b). The reference$_F$ of an expression is the object it refers to, whereas the sense$_F$ is the way in which the object is given to the mind (Frege, 1892b), or in other words, it is the thought ('Gedanke') expressed by the expression (or representation).

Consider, for instance, the two expressions '4=4' and '2+2=2·2'. The expression '2+2=2·2' is informative, in contrast to the expression '4=4'. The two expressions '2+2' and '2·2' express different thoughts but have the same reference$_F$, the natural number 4. The upshot of this; senses$_F$ capture the epistemological significance of expressions. Indeed, the algebraic structure consisting of the set of natural numbers equipped with the arithmetic operation of addition could be a possible context for both the expression '2+2' and for the expression '2·2', where multiplication would be understood as repeated addition. Notice that in this case, expressions such as '3·5' and '5·3' may be understood as different operations, because the former means 'adding five three times' while the latter is 'adding three five times'. However, there is another possible context for the expression '2·2': the algebraic structure of the set of natural numbers equipped with the arithmetic operation of multiplication. In this case, the epistemological significance of the same expression '2·2' would be different, as it would represent an operation per se, which is commutative. Thus, expressions express different thoughts concerning the different contexts where they are used. Similarly, Arzarello, Bazzini, and Chiappini (2001) called this the 'contextualized sense of an expression' that is, "a sense which depends on the knowledge domain in which it lives" (p. 63). These ideas are used as a way of recovering one of Frege's decisive insights: what sense_F comes into being is itself dependent on the context in which an object actualizes. That is, context is constitutive for sense_F.

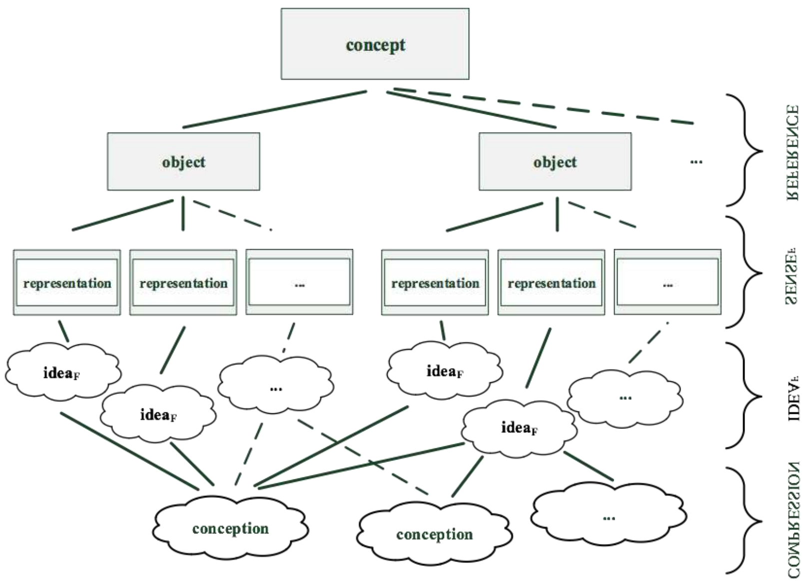

Figure 1: On reference_F, sense_F, and idea_F, (reproduced from Scheiner, 2016, p. 179)

From this position, it seems to follow that we may understand Frege's notion of an idea_F the manner in which we make sense_F of the world. For instance, one might attach the idea_F of repeated addition to the notion of multiplication. Ideas_F can interact with each other and form more compressed knowledge structures, called conceptions. For instance, one might construe '2+2 being equal to 2·2' as 'adding twice a number is the same as multiplying this by two', whereas one might construe '2·2 being equal to 2+2' as 'multiplication is repeated addition'. Alternatively, focusing on the sum and product, instead of the addition or multiplication, the sum '2+2' is equal to the product '2·2'. A general outline of the relations between concept, objects (the references$_F$ of representations), representations (expressing senses$_F$), ideas$_F$, and conceptions is provided in Figure 1.

ON CONTEXTUALIZING, COMPLEMENTIZING, AND COMPLEXIFYING

In acknowledging Frege's (1892a, 1892b) assertions, Scheiner (2016) argued that a concept does not have a fixed meaning. Rather, the meaning of a concept is relative (a) to the senses$_F$ that are expressed by representations that refer to objects falling under a concept and (b) to an individual's system of ideas$_F$. In the following, three processes are outlined that are considered to be critical in mathematical concept formation: contextualizing, complementizing, and complexifying.

Contextualizing: the epistemological function of particularizing senses$_F$

In Frege's view, a sense$_F$ can be construed as a certain state of affairs in the world and an idea$_F$ in which we make sense$_F$ of the world. Here, we started from an understanding of sense$_F$ as not primarily dependent on a mathematical object, but as emerging from the interaction of an individual with an object in the immediate context. That is, a sense$_F$ of an object at one moment in time can only be established in a more or less definite way when the process of sense$_F$-making is supported by what van Oers (1998) called *contextualizing*. Van Oers (1998) argued for a dynamic approach to context that provides the "particularization of meaning" (p. 475), or more precisely, the particularization of a sense$_F$ that comes into being in a context in which an object actualizes.

Consider, for instance, the object $\frac{3}{4}$. There are many different ways of bringing to mind $\frac{3}{4}$, even within a particular representation system (e.g., as an iconic representation as illustrated in Figure 2a and Figure 2b). Different thoughts can be expressed in different contexts: Figure 2a expresses the thought 'part of a whole' (via dividing a whole into four equal parts and directing mind to three of these four parts), whereas Figure 2b expresses the thought 'part of several wholes' (via taking three wholes, each divided into four equal parts, and directing mind to one part of each whole).

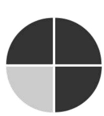

Figure 2a: Part of a whole

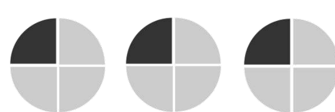

Figure 2b: Part of several wholes

Recent research suggests that individuals seem to reason and make sense_F from a specific perspective (see Scheiner & Pinto, 2018). It might be suggested that individuals take a specific perspective that orients their sense_F-making, or more accurately: in taking a particular perspective, individuals direct their attention to particular senses_F. Contextualizing, in this view, means taking a certain perspective that calls attention to particular senses_F. Attention in such cases, however, may not involve an attempt to 'sense' or 'see' anything, but it seems to be attentive thinking: attention as the direction of thinking (see Mole, 2011). As such, calling attention to particular senses_F, then, means directing mind to sense_F. In this respect, contextualizing is intentional: it directs one's thinking to particular senses_F.

Complementizing: the conceptual function of creating conceptual unity

Frege (1892b) underlined that a particular sense_F "illuminates the reference_F [...] in a very one-sided fashion. A complete knowledge of the reference_F would require that we could say immediately whether any given sense_F belongs to the reference_F. To such knowledge we never attain." (p. 27). This is to say, that just from sense_F-making of one representation that refers to an object, we are typically not in a position to know what the object is (see Duval, 2006). As contextualizing serves to particularize only single senses_F of a represented object, the same object can be 're-contextualized' (see van Oers, 1998) in other ways that support the particularization of different senses_F of the same object. Notice that senses_F can differ despite sameness of reference_F, and it is this difference of senses_F that accounts for the 'epistemological value' of different representations. It is the diversity of senses_F that has 'epistemological significance' and forms conceptual unity (see structuralist approach, Scheiner, 2016), not the similarity (or sameness) of senses_F (as might be advocated in an empiricist view). This means, what matters is to coordinate diverse senses_F to form a unity, a process called complementizing. However, the notion of 'complementizing' might be misunderstood as accumulating various senses_F (until an individual has all of them); this is not the case. Complementizing means to coordinate different senses_F to create conceptual unity.

Consider, once again, the object $\frac{3}{4}$. The two different thoughts of 'part of a whole' and 'part of several wholes' as expressed by the two different ways the object can be brought to mind are coordinated into a single unified way of presentation (see Figure 3).

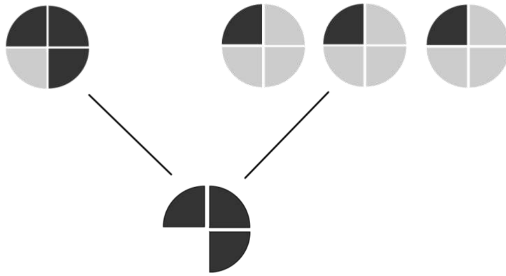

Figure 3: A conceptual unity of ‘part of a whole’ and ‘part of several wholes’

As each idea$_F$ is partial in the sense of being restricted (in space and time) and biased (from a particular perspective), it needs to be put in dialogue with other ideas$_F$ that offers an epistemological extension. The function of complementizing, then, is extending the epistemological space of possible ideas$_F$. Complementizing as extending the epistemological space of possible ideas$_F$ brings a positive stance, indicating that seemingly conflicting ideas$_F$ can be productively coordinated in a way such that these ideas$_F$ are cooperative rather than conflicting. Hence complementizing is the ongoing expansion of one’s epistemological space, the ever-unfolding process of becoming capable of new, perhaps as-yet unimaginable possibilities.

Complexifying: the cognitive function of creating a complex knowledge system

It is not only creating a unity of diverse senses$_F$, but creating an entity in its own right that forms a ‘whole’ from which emerges new qualities of the entity. That is, rather than treating the unity as a collection of different senses$_F$ that can be assigned to objects that actualize in the immediate context, it is the forming of the unity that emerges new senses$_F$ that might be assigned to potential objects.

For instance, with respect to the object $\frac{3}{4}$, the two different thoughts of ‘part of a whole’ and ‘part of several wholes’ cannot only be coordinated into a single unified way of presentation (see Figure 3), but also be blended so that it might promote the emergence of a new idea$_F$ such as, for a given sequence of entities (e.g. balls), three entities are marked and one is left out respectively (see Figure 4). Put differently; every fourth entity is not in the focus of one’s attention.

Figure 4: Sequence of three colored balls and one non-colored ball

In forming a unity, senses$_F$ are not merely considered as the parts of the unity, but “they are viewed as forming a whole with distinct properties and relations” (Dörfler, 2002, p. 342). It is, therefore, not an unachievable totality of senses$_F$ (or ideas$_F$) that matters, but how senses$_F$ (or ideas$_F$) are coordinated that develop emergent structure. This brings to the foreground a critical function of complexifying that has not been attested yet: blending previously unrelated ideas$_F$ that emerge new dynamics and structure (for a detailed account of conceptual blending, see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called a cross-space mapping, between frames from established domains (known as inputs), in order to project selectively from those inputs into a novel hybrid frame (a blend), comprised of a structure from each of its inputs, as well as a unique structure of its own (emergent structure). This strengthens Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

CONCLUSION

The emerging interpretive possibilities in thinking about contextualizing, complementizing, and complexifying have implications for theoretical, conceptual, and philosophical considerations in cognitive psychology in mathematics education. On the one hand, these perspectives call attention to a new understanding of mathematical concept formation: mathematical concept formation does not so much involve the attempt to recognize a previously unnoticed meaning of a concept (or the structure common to various objects), but rather a process of ascribing meaning to the objects of an individual's thinking from the perspective an individual has taken. That is, meaning is not so much an inherent quality of objects that is to be extracted, but something that is given to objects of one's thinking. On the other hand, in contrast to Frege (1892b), who construed a sense$_F$ in a disembodied fashion as a way an object is given to an individual, it might be suggested that individuals assign sense$_F$ to object. One is now in a position to interpret that what sense$_F$ is assigned to an object is related to what ideas$_F$ is activated in the immediate context. Recall the previous construal of Frege's notion of idea$_F$ as a manner in which an individual makes sense$_F$ of the world: ideas$_F$, it can be asserted then, orient forming the modes of presentation under which an individual refers to an object.

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