Constructing Meanings of Mathematical Registers Using Metaphorical Reasoning and Models

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Current debates about successful mathematics pedagogy suggest that mathematical learning and problem solving can be enhanced by using metaphors as they provide students with a tool for thinking. But assisting pre-service teachers to understand the importance of careful and accurate explanations for mathematical concepts remains an issue. This paper investigates how a mathematics teacher made use of models and metaphors to construct mathematical meanings within a transformational shift between less- and more-mathematical language. The Peircian model of semiosis was employed to identify the conceptual relationships in the metaphors and to analyse possible discrepancies between the literal meaning of metaphors, the teacher's intended meaning and the targeted mathematical concepts. The findings indicate that the syntactic, semantic, and pragmatic features of the language used in mathematics teaching play a significant role in student learning. Teachers' knowledge of students' prior understanding of mathematical meaning of related concepts and their knowledge of examples, models, and language that are pedagogically preferable jointly affect the quality of teaching.

Keywords: mathematical metaphors • mathematical models • mathematics and language • mathematics teacher education • models of semiosis

Mathematical Metaphors and Models

Learning to talk and think mathematically is essential for all children learning mathematics in schools. The language that teachers use in explaining mathematical concepts to students is a crucial factor in the success or failure of student learning. Thinking mathematically requires the use of specific mathematical registers that, as Chapman (1993) noted, are used to construct and communicate meaning:

> … school mathematics is a practice in which meanings are made. Teachers and learners communicate with each other in ways that are characteristic of the mathematics classroom. They follow conventional routines for doing and saying mathematics. This is the social context within which school mathematics happens. (Chapman 1993, p. 1)

Metaphors and models are frequently used by mathematics educators, often to explain new mathematical concepts in terms of already existing concepts (Carreira, 2001; Lakoff, 1994; Presmeg, 1998). The educational power of metaphor lies in its use in helping students construct abstract mathematical concepts and procedures which are difficult to represent without concrete analogies (English, 1997). However, due to the fact that teachers' use of models and metaphors sometimes draws on very private, personal images that are ripe with meaning for them as individuals (Presmeg, 1997), the possible risk associated with using models and metaphors in mathematics teaching is also a substantial issue in mathematics teacher education. For instance, a model and a metaphor can interpret and construct perfectly a certain mathematical concept, but the same model and metaphor can cause an entirely different interpretation of another concept, even though the concepts are related.

This paper discusses the main arguments that support the use of models and metaphors in constructing mathematical meanings within the transformational shift between less-mathematical and more-mathematical language in a classroom. In providing evidence of the educational power of models and metaphors in constructing abstract meaning within mathematics registers, the possible discrepancy between the metaphorical reasoning in the models, the intended meaning of the teachers, and the targeted mathematical concept being conveyed will be highlighted as important aspects for teacher educators to stress as they prepare new teachers to give these explanations to children. By employing the Peircian model of semiosis (Peirce, 1931) I am able to analyse how mathematical meanings expressed in mathematical registers were constructed from metaphorical reasoning. I also show how the transformational shift between less- and more-mathematical language can facilitate the construction of mathematical meanings from daily meanings, and thus assist students' capacities to think mathematically.

Linguistic Challenge in Mathematics Learning

The Peircian model of semiosis is a linguistic tool for the study of language. It allows for the study of links between meanings and metaphors in communicative settings. How can linguistics, the study of language, help increase our comprehension of the learning process and improve our techniques of teaching?

Research on the linguistic difficulties in learning mathematics and its implications in mathematics teaching has been carried out in different countries (e.g., Raiker, 2002; Duval, 2006; and Schleppegrell, 2007). These studies consistently indicate that the language of mathematics is a vital tool for student learning in view of the way that vocabulary learning and mathematical understanding are intertwined (Thompson & Rubenstein, 2000). The linguistic challenges of mathematics education were highlighted as early in 1978 by Michael Halliday in his discussion of the distinction between everyday and mathematical discourse and the development of mathematical registers. According to Halliday, mathematical registers are a set of mathematical terms that are expressed in a particular language structure that makes sense of mathematical meanings. Halliday states that "we should not think of a mathematical register as consisting solely of terminology, or of the development of a register as simply a process of adding new words" (1978, p. 195). Mathematical registers also require the use of particular meanings, styles and modes of argument. Because of the complexity of mathematical language, scholars such as Alexander (1988), Nolan (1984), and Shuard (1983), point out that misconceptions can arise from those mathematical registers that have general meanings in everyday language but more precise meanings in mathematical language contexts.

Even when terms have the same meaning in both every day and mathematical language, if the embedded mathematical concept is not thoroughly understood by a learner, further learning in more advanced mathematics could be impeded. In addition, a major feature of professional mathematical language is that when a term is used within the agreed mathematical context, it has an extremely high rate of unambiguousness (Dormolen, 1991). Since students, as newcomers to the context of mathematics, may not know that particular meaning of most of the mathematical words and expressions that are under discussion, teachers cannot restrict themselves to professional mathematical language, and sometimes must use non-professional, everyday language for conveying mathematical meaning. In such situations, if teachers and students' understandings of the words or expressions used are different from each others', especially when a word has a mathematical meaning that is different from the colloquial meaning accessible to the students, the possibility of miscommunication between teachers and students may arise and finally result in student misconceptions.

Given that mathematical concepts are to a large extent hierarchical (Raiker, 2002), understanding of the precise mathematical meanings of mathematical words is essential for the development of sound conceptual understanding and the subsequent development of mathematical thinking. Therefore, teachers should always be aware of the language they use. This suggests that inappropriate or imprecise use of spoken language could play a part in the formation of imperfect knowledge and misconceptions in mathematics (Raiker, 2002). As each subject area has its own way of using language to construct knowledge, a key challenge in mathematics teaching is to help students move from informal ways of constructing knowledge into the more academic ways that are necessary for disciplinary learning (Schleppegrell, 2007). Schleppegrell believes that students can build on their everyday language and knowledge and move toward new and more mathematical and technical understanding by being aware of the linguistic challenges that they might encounter.

To address the problems arising from such linguistic challenges, Chapman (1997) proposes that learning mathematics requires transformational shifts between "less mathematical" language and "more mathematical" language; and that mathematical meanings should be constructed within the shift towards increasing mathematical language use among learners.

The Use of Models and Metaphors in Mathematics Learning

The educational power of models and metaphors in making mathematical concepts meaningful and comprehensible is well established (e.g., Bazzini, 2001; Carreira, 2001; Lakoff, 1994; Lakoff & Nunez, 1997; Pimm, 1981, 1987). At the same time, the possible risks for initiating misconceptions among learners have also been found to be substantial issues in mathematics education (e.g., English, 1997; Krumholtz, 1988; Pimm, 1987; Presmeg, 1997; Sfard, 1994). One significant definition of metaphors is made by Lakoff (1994) and refined by Carreira (2001): both seeing metaphor as a correspondence between conceptual domains that embed "a mechanism that allow us to understand one domain in terms of another, usually more familiar or closer to our daily experiences" (Carreira, 2001, p. 264). Applying this concept to teaching and learning within the domain of mathematics allows us to see metaphors as tools for explaining or interpreting mathematical ideas and processes in terms of real-world events, involving everyday objects and processes (Bazzini, 2001; Pimm, 1981, 1987). Thus, metaphor provides a tool for students to think about mathematics in terms of familiar physical, material, and mental actions (McGowen & Tall, 2010). As English (1997) points out, the educational value of pedagogical use of metaphors lies in their use in developing an understanding of abstract mathematical concepts and procedures that are difficult to represent without concrete analogies. Thus, metaphors can play a possibly unique educational role in helping students acquire new knowledge (Petrie & Oshlag, 1998).

Similar to metaphors, the use of models is also intended to improve communication in conveying mathematical meaning (Krumholtz, 1988). Models are representations of a certain part of the real world by means of mathematical concepts (Blum, 1991; Edwards & Hamson, 1990; Galbraith, 1995; Niss, 1989). Models of conceptual systems can be expressed in a variety of representational media such as written symbols, spoken language, computer-based graphics, paper-based diagrams, graphs, or experienced-based metaphors for explaining other systems (Lesh & Guershon, 2003). Likewise, models "also make the coupling of distant topics through a characteristic operation of transfer between cognitive domains" (Carreira, 2001, p. 267). Carreira argues that the relationship between models and metaphors is so close that there is a submersed metaphor in every model. Metaphors are essential in the construction of all models. Carreira further elaborates her idea about the relationship between models and metaphors in mathematics learning.

To build up a mathematical model of a certain phenomenon requires some articulation between two conceptual domains, but to develop such interconnections there must be metaphors. The interactive links between the two domains can only be produced under the existence of a metaphor. Embedded in the metaphor are the needed ways of projecting inferences from one domain to the other. Therefore, the metaphor acts as the primordial element in the construction of models, and once in action it provides the semiotic mediating structure between two domains. The metaphor is necessary to the existence of the model (Carreira 2001, p. 267). Thus the appropriate projection of metaphorical thinking onto a model can generate new mathematical ideas from previous ideas. It also opens fundamental routes to mathematical understanding.

Although there is a substantial agreement about the positive influence of models and metaphors on mathematics learning, some researchers argue that a model or metaphor may have limitations because of inherent differences in the two domains (i.e., source and target) (Chiu, 2001) or because the model may place an over emphasis on some aspects and therefore neglect of others (Krumholtz, 1988). The effects of a model having this potential for similarity with a mathematical concept along with an ever-present potential for dissimilarity give models and metaphors their special power to structure new experience in terms of old, and at the same time must allow for the ambiguity which is an unavoidable element of mathematical symbol systems (Goldin, 1992). These ambiguities may lead to mis-mapping of elements between the two domains and may cast doubt on the validity of the analogy that is involved in the comparison of the domains. As mentioned earlier the power of metaphor in making sense of new concepts in terms of already existing concepts and knowledge (Presmeg, 1998), always involves some risks such as inadequate previous knowledge of learners, or a mismatch of the everyday knowledge of students and teacher. It is for this reason that Sfard (1994) pointed out that efforts to introduce appropriate metaphors are often rewarded with limited success.

Thus, although metaphors can serve as powerful means for conceptualising mathematics, metaphors may contain pitfalls for teaching and learning in cases where the intended mathematical relationships are difficult to discern, the students have difficulty in interpreting the model and metaphor themselves, or the intended meaning of the model and metaphor being delivered do not match their literal meanings (Presmeg, 1997; English, 1997). In other words, teacher use of metaphors cannot be assumed to be a completely supportive mechanism for enhancing student learning. Unless a teacher is aware of what students bring with them when they enter the classroom to learn mathematics, he or she will be unaware of what aspects of previous experience and knowledge are supportive for student learning and what aspects of previous experience may be problematic (McGowen & Tall, 2010). Thus, Krumholtz (1988) encourages teachers to be familiar with appropriate models and metaphors for the particular students in the classroom in order to use them effectively, and to employ them according to their students' cognitive level.

The Study

As noted earlier, this study investigated how a mathematics teacher constructed mathematics registers by using models and metaphors in a Cantonese-speaking classroom. The discussions and results highlight, firstly, the potential for discrepancies between the meanings of models and metaphors, the intended meaning of the teachers and the targeted mathematical concept being conveyed, and secondly, the transformational shift between less-mathematical and more-mathematical language in the teacher's discourse.

Method

In this study, I attempted to understand a secondary school mathematics teacher's use of model and metaphors in a Cantonese-speaking mathematics classroom by employing discourse analysis. This involved examining the everyday talk and explanation that arose from the construction of mathematical concepts that were embedded in mathematical *registers*, where *discourse* refers to how knowledge is constructed and shared (Ball, 1997). Gee and Green (1998) further extend the concept to include the ways of representing, thinking, talking, agreeing and disagreeing that are characteristic of a discipline or domain of knowledge. It was therefore an appropriate method of analysis for research that was focused on the contextualised talk emerging from mathematics lessons (Raiker, 2002). The teacher's discourse was interpreted in the way that a focus was put on the pattern that the teacher used to deliver the mathematical concepts and message and the communicative strategies (Schiffrin, 1994).

The teacher who participated in this study was a mathematics teacher who had been teaching in a government-aided Chinese secondary school for over twenty-five years. One double lesson and a single lesson of the teacher teaching two different topics in Grade 7 were video-taped. Details of the lessons are listed in Table 1.

Table 1
Details of the Lessons Being Studied

Number of lessons (Time)	Level/Number of students	Topic	Strand
Double lesson (60 minutes)	Grade 7/40	Similar	Geometry Triangles
Single lesson (30 minutes)	Grade 7/40	Circumference and Arc	Measurement

In order to provide greater insight into the way the teacher attempted to connect new mathematical concepts to the existing knowledge of students, the lessons chosen for recording and analysis were ones in which the teacher introduced new concepts or mathematics registers. A stationary video camera was placed at the back of the classroom throughout the lessons. Only the teacher and the blackboard writing were video recorded, in line with appropriate concerns for research ethics and issues of child protection. Full lesson transcripts in Cantonese were produced for data analysis. When the data analysis was nearly finished, a semi-structured interview with the teacher was conducted in order to assist in understanding his practice of language use in the mathematics classroom and to clarify his meanings and use of some everyday terms, which were employed for mathematical purposes in his lessons.

Framework for Analysing Mathematical Discourse

In this study, a Peircian model of semiosis (Peirce, 1931) was chosen because, as Carreira notes:

> Drawing on a semiotic perspective on meaning making and on a conceptual view of metaphor, the theoretical framework [Peircian model of semiosis] provides an outline of some possible links between meaning and metaphor and a discussion of the relation between metaphors and mathematical models (Carreira, 2001, p. 263).

Peirce's triangular model comprises three components: primary object; representamen (called a signifier in Presmeg, 1997); and interpretant. The primary object is what the sign represents or what it is taken for (Carreira, 2001) and exists independently of anything else (Presmeg, 1997). The representamen consists of the perceptible part of the sign (Carreira, 2001) and involves a relation between the primary object and some symbol of it (Presmeg, 1997). The interpretant is the element that makes the sign mean something to a particular individual in a particular context (Carreira, 2001).

Contemporary research on metaphorical reasoning interprets the notion of metaphor in a broader sense than the traditional, literary interpretation (English, 1997) and argues that the way of analysing the use of metaphors should not reside in words but in the use of words as a representation of thought (Lakoff & Nunez, 1997). In metaphors, the imagination is actively employed in reasoning about and interpreting structures and symbols (Presmeg, 1997). One potentially rich approach to the study of signs, as well as activity with signs in sign mediated processes, can be found in the field of semiotics (Presmeg, 1997). Therefore, by using Pierce's model, the study aimed at understanding the relationships among the signs (in the models and metaphors), meanings, and interpretation. This was done by identifying the targeted mathematical concept in a mathematical register (i.e., the primary object); the metaphorical reasoning embedded in the model (i.e., the representamen); and the intended meaning that the teacher intended to convey (i.e., the interpretant).

As researcher, I sought to understand how the teacher made use of models and metaphors to construct concepts in a mathematical register that would support the transformational shift between his use of less-mathematical and more-mathematical language. The recording and analysis was conducted by the researcher first in Cantonese, and then translated into English. While this is an acknowledged limitation on analysing terms, the researcher's fluency in both languages allowed a high quality translation that loses none of the salience of the example.

Results and Discussion

Below are two illustrations of the way the teacher used metaphors to develop students' understandings of the mathematical meanings of similarity and circumference.

Constructing the Mathematical Meaning of "Similar"

"Similar" is a word that has multiple meanings: a general meaning in daily life and a more precise meaning in mathematics. In his lesson discussing similar triangles, the teacher used images in photographs and in concave/convex mirrors for distinguishing and constructing the everyday meaning and mathematical concept of similarity. The following section illustrates how the mathematical use of "similar" was introduced and how its mathematical meaning was developed with the help of a model and metaphors.

The mathematical register of the word, "similar" was the key concept in this lesson. In daily life, we claim two figures are similar merely from our subjective perception and without any quantitative measurement. For instance a mushroom and a circus tent could be described as similar because they look alike in some way, such as both having round tops. In mathematics, the definition of "similarity" of figures is more rigorous. Two geometrical objects are similar only when one is congruent, with the result of a uniform scaling (enlarging or shrinking) of the other. In particular, two criteria have to been satisfied for triangles to be claimed to be similar. One criterion is that their corresponding angles have to be equal. Another is that all the corresponding pairs of sides have to be proportional. In the following two episodes (Episodes A and B) of transcribed classroom talk, I show how the everyday meanings and the mathematical meanings of "similar" were introduced in a geometry lesson.

Episode A: Distinguishing the daily and mathematical meanings of "similar".

Teacher: Okay, well, we, first, [discuss the term] similar. [For] this term alike, we always come across in our daily life. For example, the image of [your] photographs is 'like' you. Or, when you look into a convex/concave mirror, your image, of course, is not 'similar' to you. We judge 'alike' in this way. It is easy to tell whether these [things] are 'alike' or 'not alike'.

But, in Mathematics, we cannot merely depend on our perception. That means you cannot tell [they are] 'alike'—then they are 'alike'—and [they are] 'not alike'—then they are not. It is the same as to prove congruent triangles. [We] need to list some criteria [to tell] what is meant by 'like'. …

Episode B: Constructing a mathematical meaning of "similar".

Teacher: Good, through your [discussions], it is good that you can find out [the criteria]. Now, we need two criteria to prove whether figures are similar. Firstly, what do we look at? Can the angles be changed? The angles cannot be changed. When you take a photograph [of a man] and that man in the photograph is so small. But anyway, the image still looks like the man. It is because the angles [of the image in the photograph] remain unchanged. For example, a pupil with circular face, the image of his photograph still has a circular face. If his face becomes sharp, you will say that the image is not [him]. It isn't like [him]. In a convex/concave

> mirror, the image is not like [anyone]. It is because the angles are changed. [The object] is short but the image becomes lengthened [in a convex/concave]. Understand?

In episode A, references to photographs and concave and convex mirrors (both of which the teacher understood were well known to his pupils) were employed as models to introduce the everyday meaning of "similar" figures/objects. The teacher recognised that it would be straightforward to say that the images in photographs are similar to real objects while those in concave/convex mirrors are unlike the real objects. The teacher emphasised the subjectivity of our sense of perception in determining the similarity of objects in daily life. In contrast to the everyday meaning, two criteria for fulfilling the definition of "similarity" of triangles in mathematics ("the size of the object replicated can be different" but "the angles remain unchanged") are introduced to make its mathematical meaning more objective and rigorous. With the help of models, the teacher reinterpreted the literal meaning of similarity in relation to figures from daily life to mathematics; from a vague definition ("your photograph is 'like' you") to a more rigorous, more mathematical, definition ("the angles cannot be changed").

In episode B, the teacher continued to use the models to construct one of the criteria for similarity of triangles and projected a conceptual metaphorical reasoning onto the models with the analogies as shown in Figure 1 and Figure 2.

Images in photos are similar to real objects.	Their corresponding angles remain unchanged.	Two triangles in different size are similar.	Their corresponding angles are the same.

Figure 1. Metaphorical analogy for similar triangles.

Imagines in concave/convex are not similar to real objects.	Their corresponding angles are changed.	Two triangles of different shape are not similar.	Their corresponding angles are not the same.

Figure 2. Metaphorical analogy for dissimilar triangles.

The teacher emphasised the non-distorted shapes of images in photographs and justified the attribution of similarity by the criterion: corresponding angles are equal and therefore, the images are always like the real objects. On the contrary, the images in concave/convex mirrors are unlike the real objects because the angles have been changed. The teacher intended to convey the concept that any change of angles in the images would distort the original shape and therefore, the new image would be unlike the original. The mathematical metaphors in a semiotic model are shown in Table 2.

Table 2
Mathematical Metaphors for Constructing Mathematical Meaning of "Similar" in a Semiotic Model

Primary object	Representamen/Signifier	Interpretant
Similar triangles	The images in photographs are like the real objects because the corresponding angles in the images are kept unchanged.	Two triangles are similar when all corresponding angles are equal.
	The images in concave or convex mirrors are unlike the real objects because the corresponding angles in the images are changed.	Two triangles are not similar when the corresponding angles are not equal.

The analysis of these two pieces of teacher talk shows that the teacher introduced the metaphoric content in order to provide students with a platform to understand a mathematical idea: the mathematical meaning of similar. These metaphoric items signal to students the theme of the task, and a way to approach it (Chapman, 1997). The metaphoric content reshaped a daily social context to a mathematics context in a particular situation and produced a mathematical meaning which is different from its meaning in a daily context. Thus the teacher transformed the task and his language from less-mathematical to more-mathematical.

With the use of models and metaphors, the teacher tried to make the concept more visible to students. It appears from this analysis that the teacher's use of model and metaphor was able to help students capture one of the properties of similar triangles. However, when we study the mathematical definition of similarity of geometrical objects more carefully, this teacher's use of metaphor and his explanation did not construct accurately and appropriately its mathematically literal meaning. This is because the underlying mathematical concept of similarity between images in photographs and real objects is one kind of geometric transformation, the dilation of two-dimensional shapes. For further discussion of the discrepancy between the teacher's intended meaning and the literal meaning of similarity, it is necessary to understand the mathematical concepts of dilation and similarity.

The definition of dilation. Dilation is a similarity transformation in which a figure is enlarged or reduced using a scale factor, without altering the centre. The centre is a fixed point in the space where the figure lays. When the scale factor is larger than one, this is called an enlargement. When the scale factor is less than one, this is called a reduction.

Figure 3 shows an example of a dilation: triangle ABC is enlarged to triangle $A'B'C'$ with O as the centre.

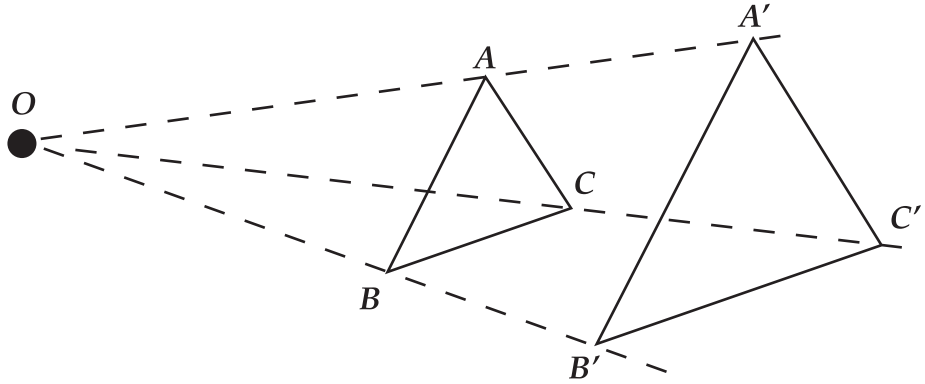

Figure 3. Dilation of a triangle

The definition of similarity of triangles. Given a correspondence between two triangles, if the corresponding angles are congruent, and corresponding sides are proportional, then the correspondence is called a similarity; and the triangles are said to be similar.
It is clear that two similar triangles are not necessarily dilations of each other. For instance, the triangles can be laid in two different spaces, or with angles pointing to different directions, as demonstrated in Figure 4.

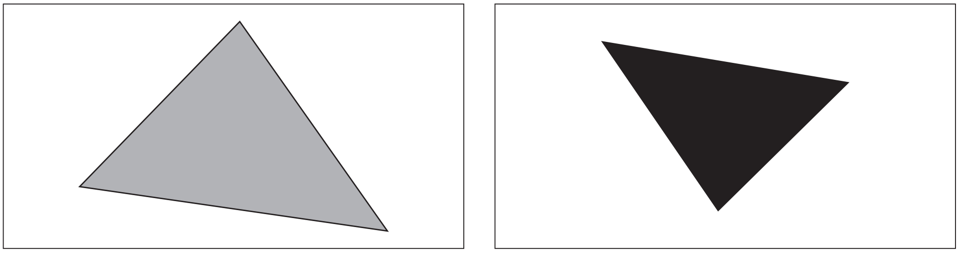

Figure 4. Example of similar triangles that are not dilations of each other.

According to the definition of *dilation*, the image in a photograph is a reduction dilation of the object. In other words, the literal meaning of the metaphor in the model used by the teacher did not match with the teacher's intended meaning. Although there was a discrepancy between the literal meaning and intended meaning, apparently it did not cause too much confusion to students' learning since the image of a triangle under dilation transformation is always similar to its pre-image. Nevertheless, to a certain extent, the prevalence of metaphorical explanations such as this in introducing the mathematical concept of similarity might explain why some students do not recognise similar triangles with different orientations. In order to provide another example of this issue for teacher learning to teach mathematics, I now turn to another example.

Constructing the Mathematical Meaning of "Circumference"

Unlike the term "similarity", which does not have congruent meanings in daily life and mathematics, the term "circumference" has a consistent meaning in both mathematical language and everyday language. In the lesson introducing the concept of circumference, the teacher made use of a very familiar model, a clock, to develop this mathematical concept. The following extract (Episode C) shows his explanation.

Episode C: Defining the mathematical meaning of "circumference".

Teacher: [Pointing at the radius and circumference of a circle on the blackboard] Or, we say, if [we] treat this radius as [the] minute hand of a clock, [any] points on this circumference is the end of the hand. When it walks a complete [cycle], [it means] walking 360º, we call it a revolution. After walking 360º, [it] means passing through a revolution, then [it] goes back to its origin. The path it passes through forms a circumference.

In this teacher's talk, a clock was employed as a model and the metaphorical reasoning was projected by the analogy shown in Figure 5.

Minute hand of a clock	The path that the end of the minute hand portrays from a point back to its original point is the outside boundary of the clock.	Radius	The path that the end of the radius portrays from a point back to the original point is a circumference of a circle.

Figure 5. Metaphorical analogy for circumference.

The teacher treated the clock as a circle and the minute hand as its radius. He emphasised that the points on the circumference were the points that the minute hand pointed to, and the circumference of the circle was the path the minute hand passed through a revolution. Table 3 shows the mathematical metaphor in a semiotic model.

Table 3
Mathematical Metaphors for Constructing Mathematical Meaning of "Circumference" in a Semiotic Model

Primary Object	Signifier	Interpretant
Radius	Minute hand of a clock.	A straight line joining the centre of a circle.
Circumference	The outside boundary of a clock.	The circumference of a circle is the locus of the radius when one end is fixed at a point and its other end rotates from one point back to the original point.

In Episode D, the teacher reinforced the mathematical definition of 'circumference' by describing his action of drawing a circle. He tied a chalk on one end of a string and fixed the other end of the string at a certain point on the blackboard. Then he moved the chalk through a revolution and a circle was drawn. The teacher focused on the action of portraying a circle in which he worked out what a circumference was and therefore, the definition of circumference was made more visible to students.

Episode D: Visualising the mathematical meaning of "circumference".

> Teacher: In the beginning of the lesson, all of you did watch me drawing a circumference, right? How did I draw a circle? We used a string, right? The chalk is the other end. It represents that end of a clock's hand. [Then] I just need to keep its distance fixed. Passing through, I rotated. I did it twice separately. Actually, I totally rotated 360º. Then, [I] drew a circumference.

Analysis of the above two pieces of the teacher's talk shows that the teacher's language operated mostly metaphorically. The teacher considered that the mathematical concept (i.e., the mathematical meaning of circumference) was too abstract, and therefore transformed the words "circle" to "clock"; "radius" to "minute hand"; and "circumference" to "the path that a minute hand walks around a revolution". He then concluded by using a string and a chalk to draw a circle. In this excerpt, the language transformation shifted from originally mathematical (i.e., circumference) to less-mathematical (i.e. clock and its components) and then back to more- mathematical (i.e. drawing a circle).

Although the model and metaphor that the teacher used could define a circumference of a circle, some important analogies between the model and metaphors, and the mathematical concepts of a circumference were not explicitly discussed in the teacher's explanation. In addition, some key concepts for defining a circumference were also not mentioned. In particular, the teacher did not explicitly emphasise that the length of the minute hand of the clock was constant. While this could have been assumed in the model because of a general knowledge of the rigidity of the minute hand on any clock, such an assumption may not always be helpful and it could not be assumed in the second model of the chalk and string. The constancy of the radius is one of the critical features for defining the circumference of a circle. Another important feature is the centre of a circle, which in terms of this model is the point of rotation of the minute hand. When the centre (i.e., the point of rotation of the minute hand) and the radius (i.e., the minute hand) are kept constant, the locus of moving the radius through a revolution forms a circle and the boundary of the circle forms the circumference. Nevertheless, the literal meaning of the model and metaphors matched the teacher's intended meaning and the targeted mathematical concept was conveyed clearly and appropriately, except for the omitted discussion of key features of invariables of centre and radius.

However, some scholars such as Pimm (1987) caution teachers about children's arbitrary transfer of model and metaphor to related mathematical concepts that may vary in different settings. On some occasions, a model and a metaphor can interpret, and construct perfectly, a certain mathematical concept but cause an entirely different interpretation of another concept though both concepts are related. For example, the circumference of a circular or spherical geometric object such as a cylinder cannot be interpreted easily with the metaphorical model of a clock and its minute hand. The circumference of a cylinder would better be described by a metaphor of walking a path around it or winding a strip of paper closely around it such as shown in Figure 6.

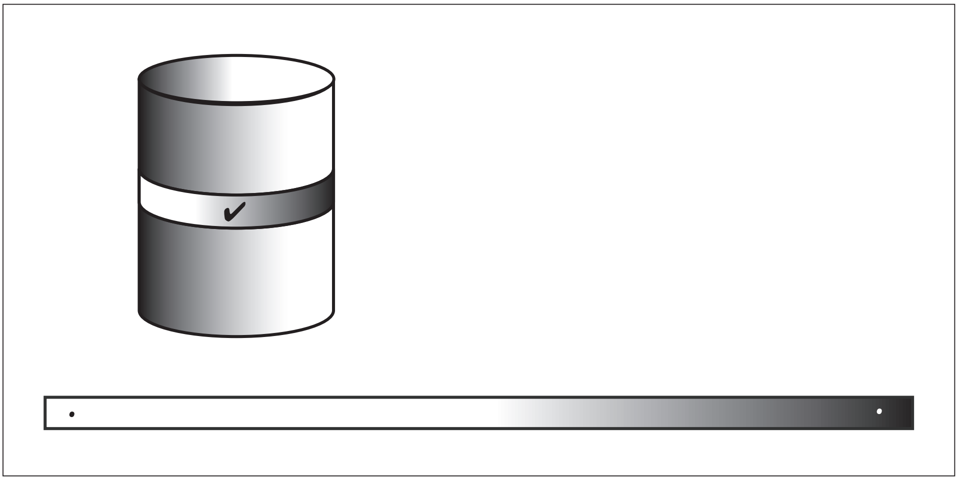

Figure 6. Winding a strip of paper around a cylinder.

A teacher could reinforce the concept using the same model by moving the strip of paper to the top or bottom rim of the cylinder and turning it on its side, to reinforce understanding of the concept of 'circumference' with reference back to the clock and minute hand metaphor.

Limitations of the Study

I recognise that the scope of this study is quite narrow, and limited to a few mathematical terms. But I consider that the conclusions and implications of this case study could be significant for mathematics teacher education. I speculate that this case study could contribute to a growing body of research dealing with teaching knowledge and the importance of developing mathematical language and understandings through use of appropriate models, metaphors, and representations. Although the study reported in this paper involves only the teaching of mathematics in Cantonese, the issues that I examine and discuss should be relevant and of interest to mathematics educators from a broad international audience. I also recognise that the study reported is "teacher focused" and therefore, acknowledge that it would be of interest to see claims supported by how the students provided evidence of their learning. But due to unresolved ethical issue at the time the study was being conducted, I was not able to measure students' achievement.

Despite the limitations as mentioned, I believe that the section concerning implications for teacher education will address some of the vital aspects of teacher knowledge, and especially the pedagogical content knowledge (Shulman, 1986) which is relevant to all in mathematics education.

An Implication for Teacher Education

Interest in the relationships between language and mathematics has remained strong since the time of Peirce's work. Over thirty years ago now, Austin and Howson (1979) listed a set of questions designed to produce illustrations of why mathematics educators should pay attention to linguistics. All of these questions remain pertinent today. One of these questions: "How can linguistics, the study of language, help increase our comprehension of the learning process and improve our techniques of teaching?" (Austin & Howson, 1979, p. 16) points to the value of linguistic analysis, such as I have carried out here, in the education of pre-service teachers.

Teachers use language to help students make connections between their own images and the concepts they meet within the language of mathematics, and thus enable them to have some control over the mathematical ideas with which they are working (Members of the Association of Teachers of Mathematics, 1991). For this reason, the development of the capacity to understand the salience of particular metaphors in the prior experience of their students and the capacity to provide teacher explanations that support the transition between less-mathematical and more-mathematical language remains a high priority in pre-service teacher education. There are two key implications here. First, teachers who have been taught and know the value of understanding the concept of circumference with reference to their own life worlds (reference to a clock, for instance, must not assume that their students will even share the same everyday understanding of 'clock' in today's digital age) must take time to understand the life worlds of their students. Second, practice in using the oral language of mathematics is vital to learning to teach mathematics because of the frequency with which verbal interpretations are required (Usiskin, 1996).

Likewise, the specific context of classroom instruction and the specific content jointly affect the syntactic, semantic, and pragmatic features of the language used in mathematics teaching, which differs from the language of everyday life not only in its explicit but also in its implicit aspects (Larborde, Conroy, Corte, Lee & Pimm, 1990). As these illustrations have shown, this teacher's use of language served important functions in passing on to students the mathematical knowledge in three interconnected senses: (i) teacher language is structured by the mathematical concepts and teacher's own educational experience; (ii) the mathematical concepts and skills students developed are largely dependent upon the teachers' use of language; and (iii) teacher language is used in the control of learning (Mousley & Marks, 1991). The use of verbal language in mathematics is important in helping students to construct new concepts, which are the basis for further learning of the written mathematical language such as symbols and proof. It is also important for teachers to be able to hear and learn from the language learners use to describe and explain mathematical concepts in relation to their own everyday life worlds.

I argue that the implication discussed aligns the central idea of Ball, Thames and Phelps's (2008) seminal work on further dividing Shulman's (1986) "Pedagogical Content Knowledge" into "Knowledge of content and students" and "Knowledge of content and teaching". In order to provide visible mathematics, teachers should be able to anticipate their students' thinking, understanding, confusions and misconceptions (Ball, et al., 2008). Developed from this idea, knowledge of content and students includes anticipating students' responses to motivate learning, and interpreting students' emerging and incomplete thinking (Ball, et al., 2008). Thus, knowledge of content and students requires the teacher's cognitive interaction between specific mathematical content understanding, familiarity with students and students' mathematical thinking (Ball et al., 2008). As noted earlier, students' prior understanding of mathematical meanings, which are the colloquial meaning of life worlds and which are different from the rigorous mathematical meaning, may give rise to misconceptions and therefore, impede their advanced mathematics learning. If teachers can anticipate (mis)conceptions and students' thinking in a specific content domain, it is speculated that students learning can be fostered. Knowledge of content and teaching refers to the knowledge that teachers require to decide how to use time in each lesson, determine the key learning points (i.e. the object of learning), choose appropriate examples, models and materials for instructional purposes, sequence the learning activities that fit students' learning path, ask appropriate questions in an appropriate order for scaffolding learning; and choose appropriate precise or unambiguous language that is pedagogically preferable for constructing concepts. Careful advance thought about such factors can make mathematics more sensible to, visible to and learnable by students. Thus, by paying special attention to students' prior knowledge and content from their perspectives (Ball & Forzani, 2009), together with choosing appropriate examples, models and language that is pedagogically preferable, it is expected that pre-service teachers' capacity for providing high quality teaching could be enhanced.

To conclude, the language factors such as metaphorical reasoning, as well as their semantic and pragmatic meanings for mathematical thought that are related to the difficulties of mathematics learning and teaching, are deserved an in-depth investigation for providing further implications for mathematics teaching and mathematics teacher education.

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