

EXPLORING YOUNG STUDENTS' FUNCTIONAL THINKING

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The Early Years Generalizing Project (EYGP) involves Australian years 1 to 4 (age 5 to 9) students and investigates how they grasp and express generalizations. This paper focuses on data collected from 6 Year 1 students in an exploratory study within a clinical interview setting that required students to identify function rules. Preliminary findings suggest that the use of gestures (both by students and interviewers), self-talk (by students), and concrete acting out, assisted students to reach generalizations and to begin to express these generalities. It also appears that as students became aware of the structure, their use of gestures and self-talk tended to decrease.

Keywords: Functional thinking; Generalization; Primary mathematics; Semiotics

Exploración del pensamiento funcional de estudiantes jóvenes

El Early Years Generalizing Project (EYGP) implica a estudiantes de primer a cuarto curso de la educación primaria australiana (de 5 a 9 años) e investiga cómo comprenden y expresan las generalizaciones. Este artículo se centra en los datos recogidos de 6 estudiantes de primer curso en un estudio exploratorio con entrevista clínica que requería que los estudiantes identificaran patrones funcionales. Los resultados preliminares sugieren que el uso de los gestos (de estudiantes y entrevistadores), las conversaciones con ellos mismos (de estudiantes), y las actuaciones concretas, ayudaron a los estudiantes a buscar generalizaciones y a comenzar a expresar estas generalidades. También parece que cuando los estudiantes tomaron conciencia de la estructura, el uso de gestos y de las conversaciones con ellos mismos tendió a disminuir.

Términos clave: Generalización; Matemáticas de primaria; Pensamiento funcional; Semiótica

EYGP is a 3-year longitudinal project that is studying a cohort of students from Year 1 to Year 4. The aim of the project is to build theories with regard to young

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students' ability to generalize mathematical structures. The cohort of students is a representation of a wide range of abilities across the first four years of school. Initially, six students from each year level participated in a one-on-one clinical interview. From the results of these interviews conjectures were posed, and were further tested in a one-on-one semi-structured interview conducted with a cohort of 20 students from each year level. This paper reports on one aspect of this project, an exploration of how 5-year old students generalize the function rule.

The concept of a function is fundamental to virtually every aspect of mathematics and every branch of quantitative science. Presently, this type of thinking is corralled at the secondary level, and yet it has many benefits for deepening younger students' understanding of arithmetic. This is particularly so in the way that operations can be considered as "changing" and how functions explicitly illustrate the way in which addition and subtraction (and multiplication and division) are inverse operations, with each "undoing" the other. Blanton and Kaput (2005) suggest that students can engage in co-variational thinking as early as kindergarten, and are capable of describing the relationship between quantities as early as Year 1. However, little is known as to whether they can reach generalizations, that is, identify the function rule and use this to predict other pairs of elements that conform to this rule, with even less being known as to how they do this. As Lannin (2005), supported by Kaput (1999) and Mason (1996) argued: "Statements of generality and discovering generality are at the very core of mathematical activity" (p. 233). Thus, this exploratory study focuses on what Radford (2006, 2010) calls the perceptual act of noticing generalities from specifics and how this occurs as 5-year old children explore the concept of a function.

Mathematics is an intrinsic symbolic activity that is accomplished through communicating using oral, bodily, written, and other signs (Radford, 2006, 2012). Semiotics lends itself to the exploration of teaching and learning activities in mathematics as this discipline is considered abstract and is heavily based on perceivable signs. Semiotics assists us to understand mathematical processes of thought, symbolization, and communication as the teaching and learning of mathematics draws on a variety of representations and resources. Two activities of particular importance to this communication is the use of the body, the activity of interacting with artifacts, and the activity with signs (Sabena, 2008). The use of the body, social, and cultural experiences is seen as strongly related to cognition (Lakoff & Núñez, 2000). It is this theoretical framework that guided our research.

In the last few years, research on generalization—focusing on upper primary to upper secondary school years—has begun to identify different approaches. Harel (2002) has proposed two different forms of generalization for mathematics induction: (a) results generalization—developing a generality from a few examples, usually by trial and error, and (b) process generalization—developing and

justifying a generality in terms that show progression across many steps. The distinction between Harel's forms appears to be very similar to Radford's (2006) distinction between naïve induction and generalization and Lannin's (2005) distinction between empirical justification and generic examples. Both Radford and Lannin's forms emerged from their studies of geometric growth patterns. Cooper and Warren (2008) also showed that 9-year old students can generalize both patterning and equivalence contexts. Incorporated in many of these theories are the notions of gesture, embodiment, and communication—including language and symbols, all considered as signs. Radford considers gestures as a type of sign, and has identified semiotic nodes as those “pieces of the students' semiotic activity where action, gesture and word work together to achieve knowledge objectification” (Radford, 2006, p. 144).

METHOD

Piagetian clinical interviews were conducted by two of the researchers with Year 1 students ($n = 6$), three female and three male with an average age of 5 years. The students were from a middle socio-economic elementary school in the outer suburbs of a major city. They were representative of a range of academic abilities and cultural backgrounds. Interviews were approximately 20 minutes in length and consisted of five tasks. Two tasks had a language focus, two a geometry (shape) focus, and one a number focus. The aim of the tasks was to probe students' understanding of functions. All questions were posed to the students in a flexible manner, and the tasks were set as play-like activities starting from unnumbered situations and moving to numbered situations. All interviews were videotaped. Table 1 presents the five tasks by focus, each tasks function rule, and an example of the input and output values for each rule presented to each student in the interview.

Table 1
Description of Tasks Used in the Interviews

| Focus | Function rule | Input values | Output values |
|----------|-----------------------------|------------------------|-----------------------|
| Language | Add “ip” | t, p, s, l | tip, pit, sit, lit |
| Language | Add “ap” | c, m, s, t | cap, map, sap, tap |
| Shape | Make it thinner | Thick red triangle | Thin red triangle |
| Shape | Make it thinner and smaller | Thick large red square | Thin small red square |
| Number | Add 2 | 3, 5, 12, 17 | 5, 7, 14, 19 |

Each task was presented using a physical function machine (Rosie) made from a cardboard box. The input and output values were presented on cards or as physi-

cal shapes. The students were introduced to the function machine, Rosie, and the interview began with the simple language task (Language 1). The student were shown a letter which they placed in Rosie's ear (input) and the researcher then produced the output card for the students from the opposite ear (output). The students were then given further examples of the rule and asked if they could describe it. The questions posed were contingent on the responses given by the students. Depending on their responses, students were either given further examples or were asked to predict output cards for given input cards. They were then asked to predict input cards for given output cards. The researcher asked students to justify their answers and express the rule and its inverse in general terms. It should be noted that these students had not engaged in formal experiences with the operations of addition and subtraction prior to this interview.

Each videotape of the interviews was transcribed with students' thinking processes documented according to their verbal responses and their manipulation of the concrete materials. All data were analyzed by at least two researchers and member checks were performed. This was particularly important with regard to identifying gestures and actions students' used as they articulated their responses. Of particular interest to this paper was students' use of gestures and self-talk as they reached generalizations and discussed their conjectures with the interviewer. In this context gestures are defined as those movements—hands, arms, eyes—that students perform during their mathematical activities (McNeill, 1992; Sabena, 2008).

FINDINGS AND DISCUSSION

For reporting purposes each student was allocated a code, namely, S1H, S2H, S1M, S2M, S1L, and S2L. The students were identified by the classroom teacher as being high achievers in mathematics (S1H and S2H), medium achievers in mathematics (S1M and S2M), and low achievers in mathematics (S1L and S2L). The data associated with each task was organized into three categories, namely, the student's ability to correctly predict: (a) output values from given input values, (b) input values from given output values, and (c) the particular function rule relating to that task. Table 2 presents the tasks together with the students who were successful in each category.

Table 2
Student's Success on the Five Tasks

| Task | Predict output | Predict input | Identify the rule |
|--------------------------------|---------------------------------|----------------------------|---------------------------------|
| Add "ip" | S1H, S2H, S1M, S2M | S2H, S1M, S1L, S2L | S2H |
| Add "ap" | S1H, S2H, S1M, S2M | S1H, S2H, S1M | S1H, S2H, S2M |
| Make it thinner | S1H, S2H, S1M, S2M, S1L, S2L | S2H, S1M, S2M, S1L, S2L | S1H, S2H, S1M, S2M, S1L, S2L |
| Make it thinner and smaller | S1H, S2H, S1M, S2M, S2L | S2H, S1M, S2M, S2L | S1H, S2H, S1M, S2M, S2L |
| Add 2 | S1H, S2H, S2M, S1L | S1H, S2H, S2M, S1L | S1H, S2H, S1M, S2M, S1L |

Preliminary findings were: (a) students' ability to identify the rule increased as they moved across the tasks, and (b) students experienced most difficulty with the language tasks as compared with the shape and number tasks. Whether this is related to the tasks themselves or as a result of student learning as they participated in investigation is uncertain. It should be noted that S2H was the only student to successfully answer all aspects of the interview.

Language played an important role in assisting students to justify their responses. For example, during Language 1 task most students experienced difficulties verbalizing Rosie's rule. While many students could identify that Rosie was changing the initial letter into a word, they experienced difficulty in describing specifically what was happening; this inability impacted on their capacity to predict the output. With the shape tasks, the majority of students experienced some difficulty due to their limited use of appropriate geometric language that would assist them to justify their answers. It appeared that as students progressed across the tasks, the sophistication of the language they used to identify the rules and to justify their predictions increased.

An example of this was the responses of S1H to the five tasks. This student's ability to identify the rule became more specific as she progressed through the tasks. She was also asked a sixth task—a number task where the rule was subtracting two. Below is an example of how this student's language became more specific in identifying the rule for each task after being asked: what is Rosie's rule?

SIH: (Response to identifying the rule for Language 1 task) nip, sip, tip, dip.

Interviewer: What does she (Rosie) do to these words? [Researcher covers up the "ip" of the word tip.]

- SIH:* She turns them into a sentence because I put the letters in and new letters came out. She turned the letters into words [SIH points to the cards]. She is doing the rules. You put this in and get it on the other side. It's magic. [SIH is constantly looking at the box. Her eyes are tracking from left to right along the function machine.]
- SIH:* (Response to identifying the rule for Language 2 task) She is making "ap" words.
- SIH:* (Response to identifying the rule for Shape 3 task): She (Rosie) turns them all flat. [During this task the researcher gestured along the front of the box from left to right. SIH manipulated the shapes in her hands and clearly described the attributes of the shape before it was placed into Rosie.]
- SIH:* (Response to identifying the rule for Shape 4 task) Two differences now. It turns it (red, big, thick triangle) into a little small one and flat again. [Once again the researcher gestured along the front of the box and the student was manipulating the shapes using a rich description.]
- SIH:* (Response to identifying the rule for Number 5 task) It is skipping one. Before I had six and it turned to eight so it skipped seven. Now I have 15 and it skipped 16 to get 17. We are skipping numbers. [SIH is now looking only at the function box and tracking her eyes. SIH is also self-talking (utterances) while determining the predictions in this task.]
- SIH:* (Response to identifying the rule for Number 6 task) It is skipping two backwards. [SIH is looking at the box and tracking with her eyes, looking at card, and using self-talk to count backwards].

In each of these cases this student not only improved in her ability to determine the generalization of each task, but also her use of gesturing and self-talk increased.

Another student that showed similar increases in gesture and self-talk was SIL, a low achieving student. Initially the gestures made by SIL were subtle; however the dynamics of these gestures and her self-talk increased as she progressed through the interview. SIL also exhibited a marked improvement in her ability to identify the rule.

- SIL:* (Response to identifying the rule for Language 1 task) Changing it. [SIL is looking at the box. Her eyes follow the box across from left to right when the examples are given. SIL does not look at the box when predicting or when identifying the rule. SIL cannot see all the output cards as the interviewer has taken them.]

- SIL:* (Response to identifying the rule for Language 2 task) She changes the little letter. She keeps a “p” on the end. [Interviewer’s hand is only moving along the back of the box. S1L predicts the input looking at the card. Student looks from left to right, turning head. Possibly reading is an issue for this student.]
- SIL:* (Response to identifying the rule for Shape 3 task) It got flatter. [S1L feels the shape as she puts in. As she states “It got flatter” her eyes follow the box and look to Rosie’s output ear. She uses her hands to describe the shape becoming flatter and to describe the triangle.]
- SIL:* (Response to identifying the rule for Number 5 task): It has to go to a higher number by 2. [S1L gesturing increases in this task. S1L moves body along the front of the box and self-talks when predicting.]
- Interviewer:* So if I put in 3 what am I going to get?
- SIL:* 5.
- Interviewer:* What did you do to work that out?
- SIL:* 3 plus 2 equals 5. [Demonstrates using fingers.]
- Interviewer:* Okay, you were right. Now this is for clever kids: What if I got 16 out? What do you think I might have put in?
- SIL:* 14. [S1L self-talking whilst solving function.]
- Interviewer:* How did you work that out?
- SIL:* I was using my toes and my hands. You take down 16 and 15 and you make it to 14. [Demonstrates using fingers and toes.]

A tentative conclusion is that students’ gesturing and self-talk in conjunction with their thinking, are important dimensions of cognitive development, and for young students this type of communication plays a vital role in bridging the gap between how they learn to think about mathematics and their conceptual understanding. In addition, this gesturing and self-talk becomes more refined as they move towards gaining this understanding. In the case of S2H this use of self-talk and gestures did not seem as imperative. S2H already seemed to have a clear understanding of the notion of function from the commencement of the interview, as indicated by his correct identification of the rule for Language 1 task.

DISCUSSION AND CONCLUSIONS

The preliminary results have identified four themes that appear to be important in assisting young students to identify the function rule, and to share their thinking of how they identified the rule.

First, the use of physical signs and the embodiment of the change process assisted these young students to associate the input value with a specific output

value and identify the rule. From a semiotic perspective these physical processes helped them to objectify each task. In all tasks, students who had the output cards placed in front of them were more successful in identifying the rule, and therefore were able to provide a generalization for the task. This was particularly important for the language tasks. However it should be noted that the students who could generalize (S1H, S2H, S2M) were all high/middle achievers.

Second, gestures/body actions from both students and researcher are an important dimension of the generalization process. Students performed better when the researcher was gesturing along the front of the box from the input to the output. With the exception of S2H, students who gestured less and did not interact with the box experienced difficulties in answering the questions posed. They also took a longer to predict outputs and explain how the function operated. In addition, these students exhibited inconsistencies in answering the questions that required prediction and appeared to be utilizing a random guess approach. By contrast, students who utilized gesturing as they engaged in the tasks experienced the most growth during the interview and were able to identify the function rule. The use of gesturing was not necessarily related to perceived cognitive ability. An example of this was S1L. For her, gesturing increased from eye movement to full body movement as the interview progressed. Accompanying this was her increased ability to identify the rule and articulate the generalization.

Third, the inclusion of concrete tactile items seemed to allow the students to explain more fully what was occurring during the functional change. This is possibly because tactile items have more tangible attributes that can be described rather than just a number or letter on a card. This was particularly evident in the geometry activities (shape tasks). Initially, when the students were given the attribute blocks they knew the shape names. Most students needed prompting during their discussion of the shapes to include attributes such as color and size. These discussions certainly appeared to assist the students identify the change rule. The geometry tasks were the two function activities in which most students experienced success.

Fourth, probing the students' initial hypothesized function rule assisted them to refine their thinking. This was achieved by directing students to test their hypothesis with other examples—provided by the interviewer and the student—or asking for further clarification using questioning techniques such as: How does Rosie do this? The researcher assisted students who did not initially identify the rule. This was achieved by encouraging them to revisit the initial examples given for the input values and to regenerate the corresponding output values.

The results of this research begin to align with Radford's notion of semiotic nodes, an idea he conjectured from research with older students. Meaning-making requires students to coordinate a range of signs as they objectify their understandings, and fundamental to this process, at the beginning stages, is the incorporation of body movement and self-talk. Even with very young students,

the use of the body—pointing, hand movement, and eye movement, and engagement with signs—the cards, the function box, and the geometric blocks—assisted these students' (the interpretant) cognitive development (Sabena, 2008). Interestingly these dimensions appear to be most important as students begin to understand the task, or concept.

This exploratory study suggests that the need of gesture and self-talk seems to diminish as the structure of the problem context becomes more apparent. Thus, as students objectify their understanding, the requirement for an array of signs, particularly concrete and iconic signs, appear to be of less importance. The question is, while S2H did not rely on gesturing and self-talk as he progressed through the tasks, how important were the other signs in assisting him to grasp the mathematical structure? Also, is there a hierarchy of signs that assist young students to reach understanding, or is it the continual mapping across signs and the bundling of signs that assist them to engage with the core understanding?

This study begins to resonate with additional results from this research, particularly in respect to young students' ability to pattern. We are beginning to hypothesize that the "act of grasping" is complex and entails two aspects: (a) the growth of identifying an underlying structure of the pattern (or function in this instance), and (b) the translation of this to a process that efficiently reaches accurate answers. In other words, a relationship between structure and efficient completion is required. This is evidenced in the data in an examination of S2L responses. By Shape tasks 3 and 4 this student is beginning to see the structure of the function, but the Number task 5 required her to also be efficient in her understanding of number. This is a dimension of mathematics where she exhibits weaknesses according to her teacher and as such she struggled with this function task. These ideas require further exploration.

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